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PARTITIONING SPACES WHICH ARE BOTH RIGHT AND LEFT SEPARATED

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0. Introduction

A space X is right separated iff X can be well-ordered in some type δ so that all initial segments are open; we say there is a right separation of X of type δ and define

$$rs(X) = \inf\{\delta : \text{there is a right separation of } X \text{ of type } \delta\}.$$

Similarly, X is left separated iff X can be well-ordered in some type δ so that all initial segments are closed; we say there is a left separation of X of type δ and define

$$ls(X) = \inf\{\delta : \text{there is a left separation of } X \text{ of type } \delta\}.$$

We say X is doubly separated iff it is both right separated and left separated. Note that there is no requirement that $rs(X) = ls(X)$.

A theorem of Gulik and Juhasz states that compact left-separated spaces are, in fact, doubly separated. In the same paper, searching for a criterion to tell which compact spaces are doubly separated, they define the concept of a vanishing sequence : $\{D_n : n < w\}$ is a vanishing sequence for X iff it partitions X and each D_n is closed discrete in $\bigcup_{j \geq n} D_j$. A compact space with a vanishing sequence is left separated. Must a compact left-separated (hence doubly

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separated) space have a vanishing sequence? Must it even have a countable partition into discrete subspaces? In fact, must a regular doubly separated space have a countable partition into discrete subspaces?

I. Nagy has shown that there is a compact left separated space which has no vanishing sequence. It is, however, a union of countably many discrete subspaces. We show that

(a) If $ls(X)$ and $rs(X)$ are small enough, where X is T_1 and doubly separated, then X is the union of countably many closed discrete subspaces.

(b) Under CH there are regular 0-dimensional doubly separated spaces which cannot be partitioned into countably many discrete subspaces.

A few more preliminaries:

Definition 1. The upper left topology on the ordinal product $\alpha \times \beta$ is the T_0 topology whose neighborhoods are all interval products $[0,\gamma] \times [\delta,\beta]$ where $\gamma < \alpha$ and $\delta < \beta$.

Characterization 2. X is doubly separated iff there is a refinement $\mathcal{J}$ of the upper left topology on $rs(X) \times ls(X)$ and a 1-1 onto function $f : rs(X) \to ls(X)$ so that X is homeomorphic to the graph of f under the relative topology induced by $\mathcal{J}$.

Proof. If $X = \{x_\gamma : \gamma < \alpha\}$ is the right separation, and $X = \{x^\delta : \sigma < \beta\}$ is the left separation, define $f(\gamma) = \delta$ iff $x_\gamma = x^\delta$.

Proposition 3. Suppose $\lambda^{<\lambda} = \lambda$, $A \subset \mathcal{P}(\lambda^+)$, $|A| = \lambda^+$, and each $a \in A$ has cardinality $< \lambda$. Then there is a $B \subset A$

with $|B| = \lambda^+$ and there is some $b \subset \lambda^+$ so that if $a, a' \in B$ then $a \cap a' = b$.

B is called a Δ-system, with b its root. Use of proposition 3 is called a Δ-system argument.

Definition 4. A family of sets is a filterbase iff every finite intersection of sets in the family is non-empty.

Definition 5. Suppose $|E| = \kappa$ and $\mathcal{A}$ is a family of subsets of E. If $A \in \mathcal{A}$, denote $A^0 = A$ and $A^1 = E - A$. Then $\mathcal{A}$ is independent iff, for any function f with domain a subset of $\mathcal{A}$ of size $< \kappa$ and with range a subset of 2, $|\bigcap_{A \in \text{dom } f} A^{f(A)}| = \kappa$. We say that $\bigcap_{A \in \text{dom } f} A^{f(A)}$ is a small Boolean combination from $\mathcal{A}$.

1. Positive Results

Throughout this section, if a space X is doubly separated we will, by characterization 2, assume it is the graph of a 1-1 function from rs(X) onto ls(X), and that the topology on X refines the upper left topology. We assume only that X is T_0.

Lemma 6. *Let* X *be doubly separated with* ls(X) = rs(X) = κ *a cardinal of uncountable cofinality. Then* X *can be partitioned into* κ *many clopen sets, each of cardinaltiy* $< \kappa$.

Proof. Let $x = \langle\gamma, f(\gamma)\rangle \in X$. Define by induction:

$$u_{x,0} = \{\langle \beta, f(\beta)\rangle \in X : \beta \le \gamma \text{ and } f(\beta) \ge f(\gamma)\}.$$

$$u_{x,2n+1} = u_{x,2n} \cup \{\langle \delta, f(\delta)\rangle : f(\gamma) \le f(\delta) \le f(\beta) \text{ for some } \langle \beta, f(\beta)\rangle \in u_{x,2n}\}$$

$$u_{x,2n+2} = u_{x,2n+1} \cup \{\langle \delta, f(\delta)\rangle : f(\delta) \ge f(\gamma) \text{ and } \delta \le \beta \text{ for some } \langle \beta, f(\beta)\rangle \in u_{x,2n+1}\}$$

Let $u_x = \bigcup_{n<\omega} u_{x,n}$. Since κ is a cardinal, each $|u_{x,n}| < \kappa$. Hence by uncountable cofinality $|u_x| < \kappa$. By the even stages of the construction each u_x is open; by the odd stages if $x = \langle \gamma, f(\gamma)\rangle$ then the boundary of u_x is contained in $\{\langle \delta, f(\delta)\rangle : \delta < \gamma \text{ and } f(\delta) < f(\gamma)\}$. We partition X into a disjoint collection of u_x's by induction:

Suppose $\{u_{x_\gamma} : \gamma < \beta\}$ is a disjoint collection with $\{f(\alpha) : \langle \alpha, f(\alpha)\rangle \in \bigcup_{\gamma<\beta} u_{x_\beta}\}$ an initial segment of $ls(X)$. Let $x_\beta = \langle \alpha, f(\alpha)\rangle$ be such that $f(\alpha)$ is minimal in $\{f(\delta) : \delta, f(\delta) \in X - \bigcup_{\gamma<\beta} u_{x_\gamma}\}$. Then $\{u_{x_\gamma} : \gamma \le \beta\}$ is still a disjoint collection satisfying the induction hypothesis, and the induction can continue until X is exhausted.

Theorem 7. *Let* X *be doubly separated, with* $rs(X) = ls(X) = \kappa^+$. *Then* X *can be partitioned into* $\le \kappa$ *many discrete subspaces. If* X *is* T_1 *the partition may consist of closed sets.*

Proof. Let $\{u_{x_\alpha} : \alpha < \kappa^+\}$ be a clopen partition as in lemma 6, each $|u_{x_\alpha}| \le \kappa$. We write $u_{x_\alpha} = \{z_{\alpha,\gamma} : \gamma < |u_{x_\alpha}|\}$ where the $z_{\alpha,\gamma}$'s are distinct. Let $D_\gamma = \{z_{\alpha,\gamma} : \alpha < \kappa^+\}$. Then there are at most κ many D_γ's, and the D_γ's partition X. Since the u_{x_α}'s are open, each D_γ is discrete. If X is T_1, each D_γ is closed.

Corollary 8. *Let* X *be doubly separated,* $rs(X) = \alpha\cdot\kappa^+$ *and* $ls(X) = \beta\cdot\kappa^+$ *where* $\alpha, \beta < \kappa^+$. *Then* X *can be partitioned* *into $\leq \kappa$ many discrete subspaces.*

Proof. For $\gamma < \alpha$ let $X_\gamma = \{x_\rho : \rho \in [\gamma\cdot\kappa^+, (\gamma+1)\cdot\kappa^+)\}$, where $\{x_\rho : \rho < \alpha\cdot\kappa^+\}$ is the right separation of X. For $\delta < \beta$ and $\gamma < \alpha$ let $X_{\gamma\delta} = \{x^\xi : \xi \in [\delta\cdot\kappa^+, (\delta+1)\cdot\kappa^+)$ and $x^\xi \in X_\gamma\}$, where $\{x^\xi : \xi < \beta\cdot\kappa^+\}$ is the left separation of X. Then $\{X_{\gamma\delta} : \gamma < \alpha, \delta < \beta\}$ partitions X into at most κ many pieces, each with rs and ls of κ^+. Apply theorem 7.

2. Counterexamples

All spaces are assumed Hausdorff.

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. We will construct a 0-dimensional, doubly separated space X with ls(X) = k,rs(X) $= \kappa^2$, and no partition into fewer than κ discrete sets. X will be constructed, as in characterization 2, as the graph of a function from a right separated space Y onto a left-separated space Z. Both Z and Y will have fairly strong properties.

Under a weaker hypothesis, the argument can be adapted to get a counterexample which is not regular, only Hausdorff. We will sketch the adaptation.

Some preliminaires: If σ is a partial function from α into 2 we write $N_\sigma = \{f \in 2^\alpha : f \supset \sigma\}$. As an abuse of notation we write dom N_σ = dom σ. The space $F(\alpha,\beta)$ for $\beta \leq \alpha$ is the set of functions 2^α under the topology whose basis consists of all N_σ, where $|\sigma| < \beta$. Note that $F(\alpha,\beta)$ is 0-dimensional.

Definition 9. (a) A space is hereditarily κ-separable if every subspace contains a dense subset of cardinality $< \kappa$.

(b) A space Y is κ-Luzin in Y^* if every nowhere dense subset of Y^* intersects Y in a set of cardinality $< \kappa$.

(c) A space has property $K(\kappa)$ if every collection of at least κ many open sets contains a subcollection which is a filterbase of size κ.

Proposition 10. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. Then there is a space Y with the following properties:

(1) $ls(Y) = \kappa$.

(2) $Y \subset F(\kappa,\lambda)$.

(3) Y is κ-Luzin in $F(\kappa,\lambda)$.

(4) Y has property $K(\kappa)$.

A theorem of Tall says that a κ-Luzin space with no pairwise disjoint family of open sets of size κ^+ is hereditarily κ-Lindelof. Thus (3) and (4) imply that Y has no discrete subspace of cardinality κ. So if $Y' \subset Y$ and $|Y'| = \kappa$ we may conclude that at most λ many elements of Y' have relative neighborhoods of cardinality $\leq \lambda$.

Proposition 11. Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. Then there is a space Z with the following properties:

(i) $rs(Z) = \kappa^2$; we identify Z as a set with κ^2.

(ii) Z is 0-dimensional.

(iii) Z is hereditarily κ-separable.

(iv) Define $Z_\alpha = \{\alpha\} \times \kappa$. Then for every basic open set $u \subset Z$ and if $Y \subset \kappa$, $|Y| = \kappa$, then $\{\alpha \in Y : Z_\sigma \subset u\}$ has size κ and $\{\alpha \in Y : Z_\alpha \subset u$ and $Z_\alpha \cap u \neq \emptyset\}$ is finite.

Proofs of propositions 10 and 11 are delayed until after the proof of the next two theorems.

Theorem 12. *Suppose* $\kappa = \lambda^+ = 2^\lambda$ *and* $\lambda^{<\lambda} = \lambda$. *Then there is a* 0*-dimensional doubly separated space* X *with* $ls(X) = \kappa$, $rs(X) = \kappa^2$, *and if* $\mathcal{D}$ *is a partition of* X *with* $|\mathcal{D}| \leq \lambda$, *then some* $D \in \mathcal{D}$ *is not discrete.*

Proof. Let Y satisfy (1) through (4) and let Z satisfy (i) through (iv). Let $f : Z \to Y$ be 1-1 onto and let X be the graph of f, under the product topology. The only non-trivial property to check is that a small partition of X contains a non-discrete set.

For $\gamma < \kappa^2$ we denote by x_γ the point $\langle \gamma, f(\gamma) \rangle \in X$. If $\mathcal{D}$ is a partition of X, $|\mathcal{D}| \leq \lambda$, then by a counting argument there are $D \in \mathcal{D}$ and $A \subset \kappa$ with $|A| = \kappa$ and if $\alpha \in A$ then $D_\alpha = \{x_\gamma \in D : \gamma \in Z_\alpha\}$ has cardinality κ. Wlog we assume $D = \bigcup_{\alpha \in A} D_\alpha$ and show it is not discrete.

For $\alpha \in A$, let $Z^*_\alpha = \{\gamma : x_\gamma \in D_\alpha\}$ and let $Y^*_\alpha = f''(Z^*_\alpha)$. We may assume that every relative neighborhood in Y^*_α has cardinality κ. By property (3) there is u_α so Y^*_α is dense in u_α. If us is an open cover of D, it has a subcollection which can be refined to the following form:

For $\alpha < \kappa$ pick $x(\alpha) \in D_\alpha$ so $x(\alpha) \in Z^*_\alpha \times u_\alpha$. Each $x(\alpha)$ is covered by $w_\alpha \times v_\alpha$ where $v_\alpha \subset u_\alpha$, and both w_α, v_α are basic in their respective topologies.

We show that $\{w_\alpha \times v_\alpha : \alpha \in A\}$ cannot be extended to a discrete cover of D.

By (4), there is $A' \subset A$, $|A'| = \kappa$, where $\{v_\alpha : \alpha \in A'\}$ is a filterbase in Y and hence in $F(\kappa,\lambda)$. By (iii) there is $B \subset A'$, $|B| \leq \lambda$ where $\bigcup_{\alpha \in B} Z^*$ is dense in $Z^* = \bigcup_{\alpha \in A'} Z^*_\alpha$. Hence by (iv) we conclude that if $\alpha \in A'$ and $\alpha > \sup B$ then, for some $\gamma \in B$, $Z^*_\gamma \subset w_\alpha$. But by (3), Y^*_γ is dense in v_γ and so there is some $\langle \beta, f(\beta) \rangle \in D_\gamma$ with $\beta \in w_\alpha$ and $f(\beta) \in v_\alpha \cap v_\gamma$; thus theorem 15 is proved.

Theorem 13. *Suppose, for some* $\kappa > \lambda$, *there is a* κ*-Luzin subspace of* 2^λ *with cardinality* κ. *Then there is a Hausdorff doubly separated space* X *with no partition into fewer than* $cf(\kappa)$ *discrete subspaces;* $ls(X) = \kappa$, $rs(X) = \kappa^2$.

Proof. Let Y be the κ-Luzin subspace of 2^λ with cardinality κ. Well order Y in type κ, $Y = \{y_\alpha : \gamma < \kappa\}$. Let $Z \subset Y^2$ so that if $\langle x,y \rangle, \langle x',y' \rangle \in Z$ then $x \neq y$ and $x = x'$ iff $y = y'$. Well order Z in type κ^2 by $<^*$. Let $f : Z \to Y$ be 1-1 so that either $f(\langle x,y \rangle) = x$ or $f(\langle x,y \rangle) = y$. Let Z_α be those elements of Z which correspond to $\{\alpha\} \times \kappa$ under $<^*$; let $Y_\alpha = f''(Z_\alpha)$. Let X be the graph of f under the following topology:

For $\langle x,y \rangle \in Z$, let u,v be disjoint open sets with $x \in u$ and $y \in v$. Let

$$B_{x,y,u,v} = \{\langle \langle x',y' \rangle, y_\gamma \rangle \in X : \langle x',y' \rangle \leq^* \langle x,y \rangle,\ \gamma \geq \delta \text{ where } f(x,y) = y_\delta, \text{ and } y_\gamma \in u \cup v\}.$$

Let the topology on X be generated by all $B_{x,y,u,v}$.

By the choice of Z this topology is Hausdorff. Again, the only non-trivial thing to check is that if $\mathcal{D}$ is a partition and $|\mathcal{D}| < cf(\kappa)$, then some $D \in \mathcal{D}$ is not discrete. Again we have A, D_α as in the previous theorem; again assume $D = \bigcup_{\alpha \in A} D_\alpha$. Again, invoke Tall's theorem. Given an open cover U of D, by another counting argument there are fixed u,v and $A' \subset A, |A'| = \kappa$ so that for each $\alpha \in A'$ there is some $B \in U$ and

$$\langle \langle x,y \rangle, f(\langle x,y \rangle) \rangle \in D_\alpha \cap B_{x,y,u,v},\ B_{x,y,u,v} \subset B.$$

But then $\mathcal{U}$ is not a discrete open cover of D.

Note that the X of theorem 13 is easily seen to be not regular.

We now turn to the proofs that the Y and Z of theorem 12 exist.

Note that any collection of discrete open sets in $F(\kappa,\lambda)$ has cardinality $\leq \lambda$. Thus if $\mathcal{U}$ is a collection of open sets and $\cup\mathcal{U}$ is dense open there is a $\mathcal{U}' \subset \mathcal{U}, |\mathcal{U}'| \leq \lambda$, and $\cup\mathcal{U}'$ is dense open. Thus to show Y is κ-Luzin in $F(\kappa,\lambda)$ it suffices to check that if $|\mathcal{U}| \leq \lambda$, and $\cup\mathcal{U}$ is dense open and each $u \in \mathcal{U}$ is open, then $|Y - \cup \mathcal{U}| < \kappa$.

Let $\{N_{\sigma_\alpha} : \alpha < \kappa\}$ list all basic open sets of $F(\kappa,\lambda)$ where $\operatorname{dom} \sigma_\alpha \subset \alpha$. Let $\{\mathcal{U}_\alpha\ \alpha < \kappa\}$ list all collections $\mathcal{U}$ of basic open sets where $|\mathcal{U}| \leq \lambda$ and $\cup\mathcal{U}$ is dense open. We say such a $\mathcal{U}$ is good for β if $N_\sigma \in \mathcal{U}$ implies $\operatorname{dom} \sigma \subset \beta$. We construct $Y = \{y_\alpha : \alpha < \kappa\}$ by induction so that

(1*) $y_\alpha \restriction \kappa-\alpha$ is identically 0.

(2*) $y_\alpha \in N_{\sigma_\alpha}$.

(3*) If $\beta < \alpha$ and $\mathcal{U}_\beta$ is good for α, then $y_\alpha \in \cup\, \mathcal{U}_\beta$.

Note that if we have a neighborhood N_σ so $N_\sigma \subset N_{\sigma_\alpha} \cap \bigcap_{\gamma<\beta} \mathcal{U}_\gamma$, for $\beta < \alpha$, then σ has an extension σ^* so $N_{\sigma^*} \subset \mathcal{U}_\beta$. By this fact the induction is completely straightforward, with the details left to the reader.

(1) is implied by (1*); (2) is trivial; (3*) implies (3); and since by a Δ-system argument $F(\kappa,\lambda)$ has property $K(\kappa)$, (2*) implies (4). Proposition 10 is proved.

Proof of Proposition 11. Some notation: If $x \in F(\lambda,\lambda)$ we say $\{x_\alpha : \alpha < \gamma\}$ converges to x iff x is in its closure and $\alpha < \beta$ implies that, for some γ, $x_\beta \restriction \gamma = x \restriction \gamma = x_\alpha \restriction \gamma$.

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. We construct a space Z as in proposition 11. The proof combines close imitations of other known constructions, so it is only sketched.

The first space imitated is the Kunen line on ω_1 [JKR] to get a space Z* which is 0-dimensional right separated hereditarily κ-separable, $rs(Z^*) = \kappa$, and each point $x \in Z^*$ has a neighborhood basis $\{\{x\} \cup \bigcup_{\beta\leq\alpha<\lambda} u_\alpha^x : \beta < \lambda\}$ where $\{u_\alpha^x : \alpha < \gamma\}$ is a disjoint family of sets clopen in $F(\lambda,\lambda)$ and there is $x_\alpha \in u_\alpha^x$ so $\{x_\alpha : \alpha < \lambda\}$ converges to x in $F(\lambda,\lambda)$. The proof is an exact imitation of the Juhasz-Kunen-Rudin construction, with κ playing the role of ω_1, λ the role of ω, and $F(\lambda,\lambda)$ the role of 2^ω.

Now switch to imitating the construction of [R]. Identify Z* with κ, preserving right separation, and construct $\mathcal{A} = \bigcup_{\lambda\leq\alpha<\kappa} \mathcal{A}_\alpha$ where each $\mathcal{A}_\alpha$ is an independent family on λ of cardinality κ so that if $B \subset \alpha$, $|B| = \lambda$, $\alpha \in$ closure B and A is a Boolean combination from $\mathcal{A}_\alpha$, then $|B \cap A| = \lambda$. $\mathcal{A}$ is constructed by a straightforward induction: at stage γ the first γ elements of each $\mathcal{A}_\alpha, \alpha \leq \gamma$, have been constructed.

Let $\mathcal{B}$ be a collection of clopen subsets of Z* so that $\mathcal{B}\{Z^* - u : u \in \mathcal{B}\}$ is a basis for Z* and if $u \in \mathcal{B}$ then $Z^* - u \notin B$. Index each $\mathcal{A}_\alpha$ by $\mathcal{A}_\alpha = \{A_u^\sigma : u \in \mathcal{B}\}$. Denote $\alpha - A_u^\alpha$ as $A_{Z^*-u}^\alpha$.

Now let Z be the following space: Z as a set is κ^2. Denote $Z_\alpha = \{\alpha\} \times \kappa$, for $\alpha < \kappa$. If $x = \langle\alpha,\beta\rangle$ then a subbasic set containing x is

$$\{\langle\alpha,\beta'\rangle : \beta' \in u\} \cup \bigcup \{Z_\gamma : \gamma \in u_\rho^\alpha \text{ for some } \rho \in A_u^\alpha - \xi\}$$

where either $u \in \mathcal{B}$ or $Z^* - u \in \mathcal{B}$, $\beta \in u$, $\xi < \alpha$. Z is clearly right separated in order type κ^2. The proofs that Z is Hausdorff and that (i) through (iv) hold are close imitations of proofs of similar statements in [R]. Thus proposition 11 is proved.

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