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PRODUCTS OF COUNTABLY COMPACT SPACES

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In this paper, we give various conditions under which a product of countably compact spaces is countably compact. In particular, we investigate the question: Does there exist a family of countably compact spaces $(X_i: i \in I)$ such that

- i) $|I| = 2^{\mathfrak{c}}$,
- ii) $\prod_{i \in I} X_i$ is not countably compact, and
- iii) if $J \subset I$ with $|J| < 2^{\mathfrak{c}}$, then $\prod_{i \in J} X_i$ is countably compact.

We give a number of consistent examples of such families, and develop several possible programs for constructing such a family in ZFC.

I. Products of Countably Compact Spaces

We begin by giving the definitions necessary for the first part of the first section.

Definitions. κ will denote a cardinal. By X^κ we mean the product of κ copies of X . A space X is sequentially compact if every sequence has a convergent subsequence. Let $\{x_n: n < \omega\}$ be a countably infinite subset of a topological space X and $x \in X$. Then x is a complete accumulation point of the set $\{x_n: n < \omega\}$ if for every open set 0 containing x , the set $\{n: x_n \in 0\}$ is infinite. In such a case, we write $x \in \{x_n: n < \omega\}'$ and we say that x is a cluster point of the set $\{x_n: n < \omega\}$.

We will use the following characterization of countable

compactness which was originally given in [AU].

Proposition. A space X is countably compact if and only if every countably infinite subset of X has a complete accumulation point.

Note that there is no T_1 requirement in the proposition and that, in particular, Theorem 2.1 which depends on it does not require a T_1 hypothesis.

The standard method of constructing $[N]$ a product of (two) countably compact spaces which is not countably compact is to construct $\omega \subset X \subset \beta(\omega)$, $\omega \subset Y \subset \beta(\omega)$, X and Y are countably compact and $X \cap Y = \omega$. Then ω (is homeomorphic to) the diagonal of $X \times Y$, and hence closed in $X \times Y$. Since ω is not countably compact, $X \times Y$ is not countably compact.

Now any sequence $(x_n: n < \omega)$ in X will have a cluster point x and a sequence $(y_n: n < \omega)$ in Y will have a cluster point y . If we want the product space $X \times Y$ to be countably compact, we have to have some way of determining when $(x, y) \in \{(x_n, y_n): n < \omega\}'$ in $X \times Y$. In particular, we need to have some way of showing that the way the sequence $(x_n: n < \omega)$ clusters at x is compatible with the way that the sequence $(y_n: n < \omega)$ clusters at y . To do this, we need to discuss the concept of \bar{D} -limit, which was first introduced in print by A. Bernstein [Be, Def. 3.1].

Definition. Let X be a space, $(x_n: n < \omega)$ a sequence in X , $x \in X$, and $\bar{D} \in \beta(\omega) \setminus \omega$. Then $x = \bar{D}\text{-}\lim_{n < \omega} x_n$ provided that if $x \in 0$ open, then $\{n: x_n \in 0\} \in \bar{D}$. If every sequence in X (not necessarily faithfully indexed) has a \bar{D} -limit point in

X , then X is \bar{D} -compact.

The important relevant facts about \bar{D} -limits are as follows.

Proposition 1.1. i) A \bar{D} -limit of a sequence of distinct points $\{x_n : n < \omega\}$ is a cluster point of the set $\{x_n : n < \omega\}$, and hence a \bar{D} -compact space is countably compact.

ii) $x \in \{x_n : n < \omega\}'$ if and only if there exists $\bar{D} \in \beta(\omega) \setminus \omega$ such that $x = \bar{D}\text{-}\lim_{n < \omega} x_n$. [GS, lemma 2.2].

iii) Let $f: X \rightarrow Y$ be a continuous map, $\{x_n : n < \omega\}$ a sequence in X and $x \in X$ such that $x = \bar{D}\text{-}\lim_{n < \omega} x_n$. Then $f(x) = \bar{D}\text{-}\lim_{n < \omega} f(x_n)$. [GS, lemma 2.3].

iv) A product of \bar{D} -limits is a \bar{D} -limit, i.e., if

$X = \prod_{i \in I} X_i$ and in each X_i , we have $x_i = \bar{D}\text{-}\lim_{n < \omega} x_i^n$, then if we define in X the points $x = (x_i)_{i \in I}$ and $x^n = (x_i^n)_{i \in I}$ for each n , then $x = \bar{D}\text{-}\lim_{n < \omega} x^n$. Therefore, a product of \bar{D} -compact spaces is \bar{D} -compact, and hence countably compact. [Be, Th. 4.2; GS, Th. 2.4].

v) X^κ is countably compact for all cardinals κ if and only if X is \bar{D} -compact, for some $\bar{D} \in \beta(\omega) \setminus \omega$. [GS, Th. 2.6].

We can now answer our earlier question of determining when $(x, y) \in \{(x_n, y_n) : n \in \omega\}'$, which follows immediately from the above facts.

Proposition 1.2. If $x \in \{x_n : n < \omega\}'$ in X and $y \in \{y_n : n < \omega\}'$ in Y , then in $X \times Y$, $(x, y) \in \{(x_n, y_n) : n < \omega\}'$ if and only if there exists $\bar{D} \in \beta(\omega) \setminus \omega$ such that $x = \bar{D}\text{-}\lim_{n < \omega} x_n$

and $y = \bar{D}\text{-}\lim_{n < \omega} y_n$.

In [Fn], S. P. Franklin discusses conditions on countably compact spaces X and Y which make the product $X \times Y$ countably compact. Proposition 1.2 allows us to state a necessary and sufficient conditions on X and Y which makes $X \times Y$ countably compact. The condition is interesting in that it has been used implicitly by several authors, most notably Frolík in [Fk₁].

Theorem 1.3. Let X and Y be countably compact spaces. Then $X \times Y$ is countably compact if and only if for every pair of sequences of distinct points $(x_n: n < \omega)$ of X and $(y_n: n < \omega)$ of Y , there exists $\bar{D} \in \beta(\omega) \setminus \omega$ such that $(x_n: n < \omega)$ has a \bar{D} -limit in X and $(y_n: n < \omega)$ has a \bar{D} -limit in Y .

This theorem can be generalized as follows:

Theorem 1.4. Let $X = \prod_{i \in I} X_i$. Then X is countably compact if and only if for every collection of sequences $\{(x_i^n: n < \omega): i \in I\}$, there exists $\bar{D} \in \beta(\omega) \setminus \omega$ such that each sequence $(x_i^n: n < \omega)$ has a \bar{D} -limit point in X_i .

Notice that in Theorem 1.4, we cannot assume that the sequences $(x_i^n: n < \omega)$ are of distinct points, as is pointed out by Eric van Douwen in the paragraph following Theorem 2.6. We are pleased to thank Eric van Douwen for some stimulating and helpful correspondence concerning some of the results discussed in this paper.

Before proceeding, let's give some definitions,

conventions and notation. If $f: A \rightarrow B$ is a function and $H \subset A$, then $f \upharpoonright H$ denotes the restriction of f to H . ω denotes the natural numbers with the discrete topology, and $\beta(\omega)$ is its Stone-Ćech compactification. If $f: \omega \rightarrow \omega$ is any function, then \bar{f} is the unique continuous function $\bar{f}: \beta(\omega) \rightarrow \beta(\omega)$ such that $\bar{f} \upharpoonright \omega = f$. If $\bar{D} \in \beta(\omega) \setminus \omega$, then

$$T(\bar{D}) = \{\bar{D}' \in \beta(\omega) \setminus \omega : \text{there exists } f: \omega \rightarrow \omega \\ \text{a bijection such that } \bar{f}(\bar{D}) = \bar{D}'\}$$

is called the type of \bar{D} in $\beta(\omega)$.

If A is a set, then as usual we have

$$[A]^\kappa = \{B \subset A : |B| = \kappa\}, [A]^{<\kappa} = \{B \subset A : |B| < \kappa\} \text{ and} \\ [A]^{<\kappa} = \{B \subset A : |B| \leq \kappa\}.$$

A sequence $(x_n : n < \omega)$ is faithfully indexed if $n \neq m$ implies $x_n \neq x_m$.

A space X is pseudocompact if every real-valued continuous on X is bounded. A space X is strongly \aleph_0 -compact if every infinite subset of X meets some compact subset of X in an infinite set.

If $\bar{D} \in \beta(\omega) \setminus \omega$, then \bar{D} is a P-point if in the space $\beta(\omega) \setminus \omega$ every countable intersection of neighborhoods of \bar{D} is a neighborhood of \bar{D} . If $\bar{D} \in \beta(\omega) \setminus \omega$, then \bar{D} is a weak-P-point if $\bar{D} \notin A'$ for any $A \in [\beta(\omega) \setminus \omega]^\omega$.

Some of the theorems in [GS] about \bar{D} -limits can be improved using recent results, and we will mention some of them now. The following result of Kunen will be particularly useful to us in this paper, in our attempt to prove theorems in ZFC.

Theorem 1.5 [K₁]. There exist 2^{\aleph_0} weak-P-points in

$\beta(\omega) \setminus \omega$ which are not P-points.

Theorem 1.6 [GS, section 5]. Let \bar{D} be a non-P-point of $\beta(\omega) \setminus \omega$. Then $T(\bar{D})$ is pseudocompact, in fact, all of the powers of $T(\bar{D})$ are pseudocompact, but no countable discrete subset of $T(\bar{D})$ has a cluster point in $T(\bar{D})$. Now, if \bar{D} is a weak-P-point which is not a P-point, then no countable subset of $T(\bar{D})$ has a cluster point in $T(\bar{D})$.

In [H, Th. 2.4 and cor. 2.5] Hechler has shown that Martin's Axiom [MA] implies that a product of no more than c strongly \aleph_0 -compact spaces is countably compact. The following theorem was originally proved in [GS] using the Continuum Hypothesis; however, it follows from Hechler's result that [MA] is sufficient.

Theorem 1.7 [MA] [GS, cor. 2.10]. If $|X| \leq c$ and if X is strongly \aleph_0 -compact, then X is \bar{D} -compact for some $\bar{D} \in \beta(\omega) \setminus \omega$. In particular, if $|X| \leq c$ and X is sequentially compact, then X is \bar{D} -compact for some $\bar{D} \in \beta(\omega) \setminus \omega$.

Various authors [K₂, JNW] have constructed sequentially compact spaces which are not \bar{D} -compact for any $\bar{D} \in \beta(\omega) \setminus \omega$ using additional axioms. To our knowledge, it still remains open whether or not such spaces can be constructed in ZFC. Theorem 1.7 shows that such a space would have to have cardinality greater than c . See [V₁] for a related construction and [V₂] for another application of \bar{D} -limits to products of countably compact spaces.

For some other theorems concerning products of countably compact spaces, see [Ri] and two recent papers by Eric

van Douwen, [vD₁] in which it is shown that [MA] implies that there exist two countably compact normal spaces whose product is not countably compact, and [vD₂] in which it is shown that [MA] implies that there exist two countably compact topological groups whose product is not countably compact.

The first precise results concerning products of countably compact spaces are due to Frolík [Fk₁], in which he proves in ZFC that $\beta(\omega) \setminus \omega$ is not homogeneous, and also that

Theorem 1.8. i) [Fk₁, Th. A] For each positive integer n, there exists a space X such that X^n is countably compact, but X^{n+1} is not; and

ii) [Fk₁, Th. B] There exists a space X such that each finite power X^n is countably compact, but X^ω is not.

The main step for all three of these results, stated in [Fk₁] in quite different terms is

Theorem 1.9 [Fk₁, Th. C]. For any $x \in \beta(\omega) \setminus \omega$, let

$F_x = \{\bar{D} \in \beta(\omega) \setminus \omega: \text{there exists a faithfully indexed discrete sequence } (x_n: n < \omega) \text{ in } \beta(\omega) \text{ such that } x = \bar{D}\text{-}\lim_{n < \omega} x_n\}.$ Then $|F_x| \leq c.$

The notion of \bar{D} -limit was introduced independently in the author's thesis in the following form: If $(x_n: n < \omega)$ is a sequence in X, and if $i: \omega \rightarrow (x_n: n < \omega)$ is the natural map taking $i(n) = x_n$, then $\bar{I}(\bar{D}) = x$. This is easily seen to be equivalent to $x = \bar{D}\text{-}\lim_{n < \omega} x_n$. Although at first glance it would appear that X needs to be completely regular Hausdorff in order to talk about \bar{I} , we show in section 5 of [S₂] that

one only needs X Hausdorff for this purpose.

II. 2^c Products Vs. 2^c Products

Before beginning this section, let's agree to the following conventions for the rest of this paper: $(X_i: i \in I)$ is a non-empty set of non-empty spaces, for $\emptyset \neq J \subset I$, we write X_J in place of $\prod_{i \in J} X_i$, and we denote by π_J the projection from X_I onto X_J . For $x \in X_I$, we use x_J interchangeably with $\pi_J(x)$.

Our point of departure is the following theorem.

Theorem 2.1 [GS, Th. 2.6; C_2 ; S_2 , Th. 2.4]. *Let $X = \prod_{i \in I} X_i$. Then X is countably compact if and only if X_J is countably compact, for all $J \in [I]^{<2^c}$.*

Proof. Since the continuous image of a countably compact space is countably compact, the "only if" statement follows from the fact that π_J is a continuous function from X onto X_J .

Suppose now that X is not countably compact, so there exists a set $\{x^n: n < \omega\} \in [X]^\omega$ with no cluster point. Then for all $\bar{D} \in \beta(\omega) \setminus \omega$, there exists $f(\bar{D}) \in I$ such that the sequence $(x_{f(\bar{D})}^n: n < \omega)$ has no \bar{D} -limit in $X_{f(\bar{D})}$.

Let $J = \{f(\bar{D}): \bar{D} \in \beta(\omega) \setminus \omega\}$ and $|J| \leq 2^c$. Then $(x_j^n: n < \omega)$ has no cluster point in X_J , since it has no \bar{D} -limit for any \bar{D} , so by Proposition 1.10, X_J is not countably compact.

We want to prove that 2^c is the best possible number in Theorem 2.1. This is unknown in ZFC [C_1 , 9.3; C_3 , problem C_7 ; S_2 , section 2 and 7.1] but we will show in a number of

different ways that it is consistent.

We want to construct a family of spaces $(X_i: i \in I)$ such that $|I| = 2^c$, X_I is not countably compact, and X_J is countably compact whenever $J \in [I]^{<2^c}$. The strongest result for which we might hope is to show that X_J is countably compact whenever $J \subsetneq I$ and our Example 2.4 shows that [MA] is sufficient for this purpose.

Our first example is the simplest one available [S₂, page 82].

Example 2.2. Let $K_x = \beta(\omega) \setminus \{x\}$, for each $x \in \beta(\omega) \setminus \omega$. Then it is well known and easy to see that K_x is strongly κ_0 -compact [Fk₂, 4.4], $\prod_{x \in \beta(\omega) \setminus \omega} K_x$ is not countably compact [Fk₃, 3.8], and $|\beta(\omega) \setminus \omega| = 2^c$. Thus it suffices to show that if $J \in [\beta(\omega) \setminus \omega]^{<2^c}$, then $\prod_{x \in J} K_x$ is countably compact.

In [H, Th. 2.4 and cor. 2.5], Hechler has shown that a product of no more than c strongly κ_0 -compact spaces is countably compact. Thus if $J \in [\beta(\omega) \setminus \omega]^{<c}$, Then $\prod_{x \in J} K_x$ is countably compact. Then if we assume $c^+ = 2^c$, then $\prod_{x \in J} K_x$ is countably compact, whenever $J \in [\beta(\omega) \setminus \omega]^{<2^c}$. Thus [MA] and $c^+ = 2^c$ are enough to guarantee that $\prod_{x \in J} K_x$ is countably compact.

In section 3, we will consider an alternative approach to this goal.

Lemma 2.3. Suppose there exists a set I of types of ultrafilters on ω with the following properties:

- (i) $|I| = 2^c$,

(ii) \mathcal{D}_i is of type i for each i , and

(iii) if we set

$F_i = \{x \in \beta(\omega) \setminus \omega : \text{there exists a sequence}$

$(x_n : n < \omega) \text{ in } \beta(\omega) \text{ with } x = \mathcal{D}_i\text{-}\lim_{n < \omega} x_n \text{ and}$

$\{n : x_n \neq x\} \in \mathcal{D}_i\}$,

then $i \neq j$ implies that $F_i \cap F_j = \emptyset$. Then if we set

$X_i = \omega \cup \bigcup_{i \neq j} F_j$, then $X = \bigcap_{i \in I} X_i$ is not countably compact and if $J \subsetneq I$, then X_J is countably compact.

Proof. X is not countably compact because the condition $F_i \cap F_j = \emptyset$ guarantees that ω is (homeomorphic to) the diagonal of X . Now let $J \subsetneq I$ and there exists $i_0 \in I \setminus J$. Then for $i \in J$, each X_i is \mathcal{D}_{i_0} -compact, hence X_J is \mathcal{D}_{i_0} -compact, and so X_J is countably compact.

So the question is: Does such a set I exist? Assuming [MA], the answer is yes, using the notion of selective ultrafilter.

Definition. An ultrafilter \mathcal{D} on ω is a selective ultrafilter if whenever $\{A_n : n < \omega\}$ is a countable decomposition of ω and for all $n < \omega$, $A_n \notin \mathcal{D}$, then there exists $A \in \mathcal{D}$ with $|A \cap A_n| \leq 1$, for all n . Clearly, a selective ultrafilter is a P -point. In [Bl] A. Blass showed that [MA] implies the existence of $2^{\mathfrak{c}}$ different selective ultrafilters on ω , and in particular the existence of $2^{\mathfrak{c}}$ different types of selective ultrafilters on ω , which is the result we need.

In $[K_3]$, Kunen has shown that it is consistent that there are no selective ultrafilters. Thus selective ultrafilters cannot be used to show that the family I of Lemma 2.3 exists in ZFC.

The following notation will be useful: If $\mathcal{D} \in \beta(\omega) \setminus \omega$, $x = \mathcal{D}\text{-}\lim_{n < \omega} x_n$ and $S \subset \omega$ with $S \in \mathcal{D}$, then $x = \mathcal{D}\text{-}\lim_{n \in S} x_n$, indicating that x is a \mathcal{D} -limit of the sequence $(x_n: n \in S)$. Clearly if \mathcal{D} is a selective ultrafilter, $x = \mathcal{D}\text{-}\lim_{n < \omega} x_n$ and $\{n: x_n \neq x\} \in \mathcal{D}$, then there exists $S \in \mathcal{D}$ with $x = \mathcal{D}\text{-}\lim_{n \in S} x_n$ and $(x_n: n \in S)$ is faithfully indexed.

Example 2.4 [S_2 , Th. 2.7]. So let

$$I = \{T(\mathcal{D}): \mathcal{D} \text{ is a selective ultrafilter on } \omega\}.$$

Then [MA] implies that $|I| = 2^{\mathfrak{c}}$ and we want to show that I satisfies the hypothesis of Lemma 2.3. To this end, let $i \neq j$, \mathcal{D}_i of type i and \mathcal{D}_j of type j . Then we must show that $F_i \cap F_j = \emptyset$, that is, that if $x = \mathcal{D}_i\text{-}\lim_{n < \omega} x_n$ and $y = \mathcal{D}_j\text{-}\lim_{n < \omega} y_n$, then $x \neq y$. We will sketch the proof here, the details may be found in [S_2 , lemmas 2.5 and 2.6].

First since \mathcal{D}_i and \mathcal{D}_j are selective ultrafilters, then we may assume that the sequences $(x_n: n < \omega)$ and $(y_n: n < \omega)$ are faithfully indexed. Then since \mathcal{D}_i and \mathcal{D}_j are P-points, we may find sets A and B such that $x = \mathcal{D}_i\text{-}\lim_{n \in A} x_n$, $y = \mathcal{D}_j\text{-}\lim_{n \in B} y_n$ and the sets $\{x_n: n \in A\}$ and $\{y_n: n \in B\}$ are discrete. Actually we can maneuver so that the set $\{z_n: n < \omega\} = \{x_n: n \in A\} \cup \{y_n: n \in B\}$ is discrete. Then x is a type of \mathcal{D}_i -limit of $(z_n: n < \omega)$ and y is a type of \mathcal{D}_j -limit of $(z_n: n < \omega)$ and so $x \neq y$.

Thus Example 2.4 shows the following:

Theorem 2.5. [MA] implies there exists a family of spaces $(X_i: i \in I)$ such that $|I| = 2^{\mathfrak{c}}$, X_i is not countably

compact, and if $J \subsetneq I$, then X_J is countably compact; in particular, if $J \in [I]^{<2^c}$, then X_J is countably compact.

Question. Can the family I of Lemma 2.3 be constructed in ZFC?

We do not know, but the following recent result of S. Shelah which just came to our attention might be useful.

Theorem [Shelah, SR]. There is a collection of 2^c ultrafilters on ω which are pairwise incompatible in the Rudin-Keisler order.

Let's digress for a moment to look at a different question. Since we cannot as yet find in ZFC a family of 2^c countably compact spaces whose product is not countably compact but all of whose $<2^c$ subproducts are countably compact, let's try to determine which cardinals in place of 2^c allow such a construction. Theorem 1.8 due to Frolik shows that the finite cardinals and also ω allow such a construction. We now show that Frolik's results can be used to show that ω_1 allows such a construction. This result has been obtained, independently and using different methods, by Eric van Douwen (personal communication).

Theorem 2.6. There exists a family of countably compact spaces $(x_\alpha: \alpha < \omega_1)$ such that $\prod_{\alpha < \omega_1} x_\alpha$ is not countable compact but $\prod_{\alpha < \beta} x_\alpha$ is countably compact, for all $\beta < \omega_1$.

Proof. In Lemma 2.2 of [Fk₁], Frolik constructs a family $(Y_\alpha: \alpha < \omega_1)$ of subsets of $\beta(\omega) \setminus \omega$ and a family $(\mathcal{D}_\alpha: \alpha < \omega_1)$ of ultrafilters on ω such that

- i) $\alpha \neq \beta$ implies $Y_\alpha \cap Y_\beta = \emptyset$,
- ii) $\alpha \neq \beta$ implies ∂_α and ∂_β have different types, and
- iii) if $(x_n: n < \omega)$ is a faithfully indexed discrete sequence of $\bigcup_{\beta < \alpha} Y_\beta$, and if $x = \partial_\alpha\text{-}\lim_{n < \omega} x_n$, then $x \in Y_\alpha$.

Now let $X_\alpha = \omega \cup \bigcup_{\beta \neq \alpha} Y_\beta$.

Then $\prod_{\alpha < \omega_1} X_\alpha$ is not countably compact because $Y_\alpha \cap Y_\beta = \emptyset$.

It follows immediately from Theorem D of [Fk₁] that $\prod_{\alpha < \beta} X_\alpha$ is countably compact, for all $\beta < \omega_1$.

At one time, we thought that a stronger result could be derived from this situation, and we are grateful to Eric van Douwen for showing us an error in our proof, by pointing out that the following is not provable in ZFC: Given a sequence $X = (x_n: n < \omega)$ in ${}^\omega \omega_1$, there exists $Y \in [X]^\omega$, such that for all $\alpha < \omega_1$, there exists $F_\alpha \in [Y]^{<\omega}$, $\pi_\alpha \restriction (Y \setminus F_\alpha)$ is either constant or one-to-one.

III. Cluster Sets

Let us return to Example 2.2. Set $K_x = \beta(\omega) \setminus \{x\}$, for each $x \in \beta(\omega) \setminus \omega$. Then $\prod_{x \in \beta(\omega) \setminus \omega} K_x$ is not countably compact and we want to show that if $J \in [\beta(\omega) \setminus \omega]^{<2^C}$ then $\prod_{x \in J} K_x$ is countably compact.

The following conjecture which is consistent with ZFC is sufficient to do this; first we need a new definition.

Definition. Let $C \subset \beta(\omega) \setminus \omega$. Then C is a cluster set if there exist $x \in \beta(\omega) \setminus \omega$ and a sequence (not necessarily faithfully indexed) $(x_n: n < \omega)$ in $\beta(\omega)$ such that

$$C = \{\partial \in \beta(\omega) \setminus \omega: x = \partial\text{-}\lim_{n < \omega} x_n, \{n: x_n \neq x\} \in \partial\}.$$

Conjecture. $\beta(\omega) \setminus \omega$ is not the union of $< 2^c$ cluster sets.

Theorem 3.1. Assume that the conjecture is true. Then if $|J| < 2^c$, then $\prod_{x \in J} K_x$ is countably compact.

Proof. Let $(x^n: n < \omega)$ be a sequence in $\prod_{x \in J} K_x$. For each $x \in J$, let

$$C_x = \{ \bar{D}: x = \bar{D}\text{-}\lim_{n < \omega} \pi_x(x^n) \}$$

Then $\bigcup_{x \in J} C_x \neq \beta(\omega) \setminus \omega$ so there exists

$$\bar{D} \in (\beta(\omega) \setminus \omega) \setminus \bigcup_{x \in J} C_x.$$

Then find $y_x = \bar{D}\text{-}\lim_{n < \omega} \pi_x(x^n)$ and $y_x \in K_x$, so by letting

$y = (y_x)_{x \in J}$, we have $y \in \prod_{x \in J} K_x$ and $y = \bar{D}\text{-}\lim_{n < \omega} x^n$, and so

$\prod_{x \in J} K_x$ is countably compact.

Remarks. i) Cluster sets are nowhere dense, which follows from the fact that $\beta(\omega)$ has no convergent sequences.

ii) The conjecture is consistent with ZFC. This follows from consistency result VI of [BPS].

iii) If $(x_n: n < \omega)$ is faithfully indexed discrete, then $|C_x| = 1$ for all $x \in \{x_n: n < \omega\}'$. [Fk₁, 1.5; R, lemma 2 and 3]

iv) A cluster set also has the form $C = \bar{I}^{-1}(x)$, where $i: \omega \rightarrow \{x_n: n < \omega\}$ and $x \in \{x_n: n < \omega\}'$.

One of our early questions concerning cluster sets was whether or not they are separable. The following example of a cluster set with c isolated points, and hence not separable, is due to Eric van Douwen (personal communication) and is included here with his kind permission. It will be

convenient to use the following notation. If A and B are sets, then ${}^A B = \{f: f: A \rightarrow B \text{ is a function}\}$. If $k < \omega$, then $k = \{i < \omega: i < k\}$.

Theorem 3.2 [van Douwen]. There exists a cluster set with c isolated points.

Proof. Let $f: \omega \rightarrow \omega$ satisfy $|f^{-1}(n)| = 2^n$ for each n . There is a $\mathcal{G} \subset \mathcal{P}(\omega)$ such that $|\mathcal{G}| = c$, $|I \cap J| < \omega$ for distinct $I, J \in \mathcal{G}$ and $f \restriction I: I \rightarrow \omega$ is a bijection, for $I \in \mathcal{G}$. Let's verify the existence of \mathcal{G} . For each $n < \omega$, let $f^{-1}(n) = \{n_f: f \in {}^n\{0,1\}\}$, which is permissible since $|f^{-1}(n)| = 2^n$. Now for $s \in {}^\omega\{0,1\}$, set $I_s = \{n_{s \restriction n}: n < \omega\}$, and let $\mathcal{G} = \{I_s: s \in {}^\omega\{0,1\}\}$. If $s_1 \neq s_2$, then there exists $k < \omega$ such that $s_1(k) \neq s_2(k)$. Thus if $m > k$, $s_1 \restriction m \neq s_2 \restriction m$, so $I_{s_1} \neq I_{s_2}$, and $|\mathcal{G}| = c$. Furthermore, we have shown that $|I_{s_1} \cap I_{s_2}| \leq k < \omega$. Then we have $|\mathcal{G}| = c, |I \cap J| < \omega$ for distinct $I, J \in \mathcal{G}$ and $f \restriction I: I \rightarrow \omega$ is a bijection since $|I \cap f^{-1}(n)| = 1$, for all $n < \omega$.

Pick any $q \in \beta(\omega) \setminus \omega$. Then $\bar{f}^{-1}(q)$ is a cluster set which has c isolated points: Since $f \restriction I: I \rightarrow \omega$ is a bijection, $\bar{f} \restriction I: \text{cl}_{\beta(\omega)} I \rightarrow \beta(\omega)$ is a bijection, hence $\text{cl}_{\beta(\omega)} I \cap \bar{f}^{-1}(q)$ consists of one point; call this point x_I , then for $I \in \mathcal{G}$ x_I is an isolated point of $\bar{f}^{-1}(q)$. Also if $I \neq J$ then $\text{cl}_{\beta(\omega)} I \cap \text{cl}_{\beta(\omega)} J \cap \beta(\omega) \setminus \omega = \emptyset$, hence $x_I \neq x_J$ for distinct $I, J \in \mathcal{G}$.

Although we do not know if the conjecture is true in ZFC, we really do feel that it is. For by remark iv) a cluster set is $1/2^c$ of $\beta(\omega) \setminus \omega$ so the union of less than 2^c cluster sets should be

$$\sum_{<2^c} \frac{\beta(\omega) \setminus \omega}{2^c} = \frac{<2^c}{2^c} \beta(\omega) \setminus \omega \neq \beta(\omega) \setminus \omega.$$

IV. An Application of Weak-P-Points

We begin this section with a general lemma. We will say that a sequence $(x_n: n < \omega)$ is almost constant or almost one-to-one if there exists N such that $(x_n: n > N)$ is constant or one-to-one.

Lemma 4.1. Let $(X_i: i \in I)$ be a family of countably compact spaces such that

- i) $\omega \subset X_i \subset \beta(\omega)$, for all $i \in I$,*
- ii) there exists $\bar{D} \in \beta(\omega) \setminus \omega$ such that $T(\bar{D}) \subset X_i$, for each i , and*
- iii) if $A \in [X_i \setminus \omega]^\omega$, then $\text{cl}_{X_i} A$ is compact.*

Then a) for any $A \in [X_I]^\omega$, if $\pi_i(A)$ is almost constant or almost one-to-one on each factor i , then A has a cluster point in X_I ,

- b) if $J \in [I]^{<\omega}$, then X_J is countably compact, and*
- c) if $|I| \leq \omega$, then X_I is countably compact.*

Proof. a) Let $A = \{x^n: n < \omega\}$ be as in a), and we will define a cluster point z for A in X_I . Let $i \in I$.

If $\pi_i(A)$ is almost constant, let $z_i =$ the constant.

If $\pi_i(A)$ is almost one-to-one, we distinguish between two cases.

Case i) $G_i = \{n: x_i^n \in \omega\} \in \bar{D}$. Then we can find $f: \omega \rightarrow \omega$ which is almost one-to-one and $f(n) = x_i^n$ for each $n \in G_i$. Then $\bar{f}(\bar{D}) \in T(\bar{D}) \subset X_i$ and setting $z_i = \bar{f}(\bar{D})$, we have $z_i = \bar{D}\text{-}\lim_{n < \omega} x_i^n$.

Case ii) not case i). Then $\{n: x_i^n \notin \omega\} \in \mathcal{D}$ and so let $K_i = \pi_i(A) \cap (X_i - \omega)$.

Then set $z_i = \mathcal{D}\text{-}\lim_{n < \omega} x_i^n$ and $z_i \in X_i$ since $\text{cl}_{X_i} K_i$ is compact.

Now let $z = (z_i)_{i \in I}$ and $z_i = \mathcal{D}\text{-}\lim_{n < \omega} x_i^n$, for all $i \in I$, and so $z = \mathcal{D}\text{-}\lim_{n > \omega} x^n$.

b) Since $|J| \leq \omega$, then for any $A \in [X_J]^\omega$, there exists $B \in [A]^\omega$ such that $\pi_i(B)$ is almost constant or one-to-one $[Fk_1, \text{pf. or Th. D}]$, for each $i \in J$. Then part a) guarantees that B has a cluster point in X_J , and so A has a cluster point.

c) follows from b).

Recall that Kunen proved that there exist $2^{\mathcal{C}}$ weak-P-points in $\beta(\omega) \setminus \omega$; in particular there exist $2^{\mathcal{C}}$ weak-P-point types.

So let $\mathcal{A} = \{T(\mathcal{D}): \mathcal{D} \text{ is a weak-P-point in } \beta(\omega) \setminus \omega\}$.

Then if $t \in \mathcal{A}$, then $t \subset \beta(\omega) \setminus \omega$ so if $\beta \subset \mathcal{A}$ we set

$$\bigcup_{t \in \beta} t = \{p \in \beta(\omega) \setminus \omega: p \in t \text{ for some } t \in \beta\}.$$

Then since a countable set of weak-P-points is discrete, and any two disjoint countable sets of weak-P-points have disjoint closures, we have:

Lemma 4.2. If $x \in \text{cl}_{\beta(\omega)} A$ for some $A \in [\bigcup_{t \in \mathcal{A}} t]^\omega$ then there exists $C \in [A]^\omega$ such that $x \notin \text{cl } B$, for all $B \in [\bigcup_{t \in \mathcal{A} \setminus C} t]^\omega$.

Proof. Let $C = \{t: \text{there exists } x_n \in A \text{ of type } t\}$, and $C \in [A]^{<\omega}$ clearly works.

This allows us to construct

Example 4.3. For $Z \in [A]^\omega$ let

$$X_Z = \omega \cup \bigcup \{ \text{cl}_{\beta(\omega)} S : S \in \bigcup_{t \in A \setminus Z} t \}^\omega$$

and let $X = \prod_{Z \in [A]^\omega} X_Z$.

Then

- i) $|[A]^\omega| = 2^C$,
- ii) $\prod_{Z \in [A]^\omega} X_Z$ is not countably compact.
- iii) if $S \in [X_Z \setminus \omega]^\omega$, then $\text{cl}_{X_Z} S$ is compact,
- iv) in $< 2^C$ subproducts, every sequence whose projection onto each factor is almost constant or almost one-to-one has a cluster point, and
- v) countable subproducts are countably compact.

Proof. Lemma 4.2 guarantees that $\bigcap_{Z \in [A]^\omega} X_Z = \omega$, so that $\prod_{Z \in [A]^\omega} X_Z$ is not countably compact.

Statements i) and iii) are immediate.

To see iv) let $I \subset [A]^\omega$, $|I| = \kappa < 2^C$. Since for each $Z \in I$, $|Z| \leq \omega$, then $\bigcup_{Z \in I} Z \leq \omega \cdot \kappa = \kappa < 2^C$.

Thus we have that there exists $t \in A \setminus I$. Then $t \subset X_Z$, for each $Z \in I$, so $(X_Z : Z \in I)$ satisfies the hypothesis of Lemma 4.1, and result a) of Lemma 4.1 is exactly what we need.

v) follows from b) of Lemma 4.1.

Statement iii) of Lemma 4.2 shows that in order to prove that a sequence in $\prod_{Z \in I} X_Z$, $|I| < 2^C$ has a cluster point, we need only worry about sequences from ω . To deal with sequences from ω , we would prefer, instead of types, to take sets of the form

$$F(\mathcal{D}) = \{\mathcal{D}' \in \beta(\omega) \setminus \omega : \mathcal{D}' = \overline{f}(\mathcal{D}) \text{ for some } f: \omega \rightarrow \omega\}.$$

The problem is that we may not have an analogue of Lemma 4.2 using $F(\mathcal{D})$ instead of types $T(\mathcal{D})$. So what we need is the following:

Does there exist a family β of weak-P-points such that

$$\text{i) } |\beta| = 2^{\mathfrak{C}}.$$

$$\text{ii) if } x \in \text{cl}_{\beta(\omega)} A \text{ for some } A \in \left[\bigcup_{\mathcal{D} \in \beta} F(\mathcal{D}) \right]^{\omega}$$

then there exists $C \in [\beta]^{\omega}$ such that $x \notin \text{cl } B$, for all

$$B \in \left[\bigcup_{\mathcal{D} \in \beta \setminus C} F(\mathcal{D}) \right]^{\omega}.$$

Then the analogue of Example 4.3 with $F(\mathcal{D})$ instead of types would give us an example of a product of $2^{\mathfrak{C}}$ countably compact spaces which is not countably compact, but every $2^{\mathfrak{C}}$ subproduct would be countably compact.

Example 4.3 shows that a major source of difficulty in this problem is the fact that in big product spaces, we cannot take our sequences to be almost constant or almost one-to-one on each factor.

V. Products of \mathcal{D} -Compact Spaces for Different \mathcal{D} 's

In this section, we consider the question: Do there exist countably compact spaces X and Y such that X^{κ} and Y^{κ} are countably compact, for all cardinals κ , but $X \times Y$ is not countably compact. To our knowledge, this is unknown in ZFC, but we do have the following results. The notation will remain the same as in the previous section.

Example 5.1 [MA]. There exist spaces X and Y such that X^{κ} and Y^{κ} are countably compact, for all cardinals κ , but $X \times Y$ is not countably compact.

Example 5.2. There exist space X and Y such that X^ω and Y^ω are countably compact, and $X \times Y$ is not countably compact.

These examples may be compared with an example of E. Michael [CH] [M] of two Lindelöf spaces X and Y such that X^ω and Y^ω are Lindelöf, and $X \times Y$ is not Lindelöf. I am grateful to Jerry Vaughan for supplying this reference.

Example 5.2 has been obtained, independently, by Eric van Douwen.

Example 5.1 [MA]. Let $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$ be selective ultrafilters on ω of different types. It is not difficult to see that since $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$ are selective, then $T(\bar{\mathcal{D}}_1) = F(\bar{\mathcal{D}}_1)$ and $T(\bar{\mathcal{D}}_2) = F(\bar{\mathcal{D}}_2)$. This also follows directly from the fact that selective ultrafilters are minimal in the Rudin-Keisler order [see e.g., CN, Th. 9.6].

Then let

$$X = \omega \cup F(\bar{\mathcal{D}}_1) \cup \bigcup \{cl_{\beta(\omega)} S : S \in [F(\bar{\mathcal{D}}_1)]^\omega\}$$

and

$$Y = \omega \cup F(\bar{\mathcal{D}}_2) \cup \bigcup \{cl_{\beta(\omega)} S : S \in [F(\bar{\mathcal{D}}_2)]^\omega\}$$

We claim that X is $\bar{\mathcal{D}}_1$ -compact. Let's first verify that $X \setminus \omega$ is $\bar{\mathcal{D}}_1$ -compact. Let $(a_n : n < \omega)$ be a sequence in $X \setminus \omega$, $A = \{a_n : n < \omega\}$. Then for each n , either $a_n \in F(\bar{\mathcal{D}}_1)$ and we let $A_n = \{a_n\}$, or there exists $A_n \in [F(\bar{\mathcal{D}}_1)]^\omega$ with $a_n \in cl_{\beta(\omega)} A_n$. Then $S = \bigcup_{n < \omega} A_n \in [F(\bar{\mathcal{D}}_1)]^\omega$ and so $cl_{\beta(\omega)} S \subset X$, that is, $cl_X S$ is compact. Then $cl_X A$ is compact, and hence $cl_X A$ is $\bar{\mathcal{D}}$ -compact, for all $\bar{\mathcal{D}} \in \beta(\omega) \setminus \omega$ [Be, Th. 3.4]. Then $cl_X A$ is $\bar{\mathcal{D}}_1$ -compact and $(a_n : n < \omega)$ has a $\bar{\mathcal{D}}_1$ -limit in X . Thus $X \setminus \omega$ is $\bar{\mathcal{D}}_1$ -compact.

Now let $(x_n: n < \omega)$ be a sequence in X . If $H = \{n: x_n \in \omega\} \in \bar{\mathcal{D}}_1$, then find $f: \omega \rightarrow \omega$ with $f(n) = x_n$, for all $n \in H$. Then $\bar{f}(\bar{\mathcal{D}}_1) \in F(\bar{\mathcal{D}}_1) \subset X$, and $\bar{f}(\bar{\mathcal{D}}_1) = \bar{\mathcal{D}}_1\text{-}\lim_{n < \omega} x_n$. If $H \notin \bar{\mathcal{D}}_1$, then $K = \{n: x_n \notin \omega\} \in \bar{\mathcal{D}}_1$. Since $X \setminus \omega$ is $\bar{\mathcal{D}}_1$ -compact, $(x_n: n \in K)$ has a $\bar{\mathcal{D}}_1$ -limit point x , and clearly $x = \bar{\mathcal{D}}_1\text{-}\lim_{n < \omega} x_n$. Thus X is $\bar{\mathcal{D}}_1$ -compact. Similarly Y is $\bar{\mathcal{D}}_2$ -compact.

By construction, $X \cap Y = \omega$, so $X \times Y$ is not countably compact.

An alternative example with the same properties, using the notation of Lemma 2.3 and Example 2.4, is provided by letting $X = \omega \cup F_1$ and $Y = \omega \cup F_2$. Then X is $\bar{\mathcal{D}}_1$ -compact, Y is $\bar{\mathcal{D}}_2$ -compact and $X \cap Y = \omega$ so $X \times Y$ is not countably compact.

Example 5.2. Let $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$ be weak-P-points of different types.

Set $X = \omega \cup \bigcup \{cl_{\beta(\omega)} S: S \in [T(\bar{\mathcal{D}}_1)]^\omega\}$

and

$Y = \omega \cup \bigcup \{cl_{\beta(\omega)} S: S \in [T(\bar{\mathcal{D}}_2)]^\omega\}$

Then let $X_n = X$, for all $n < \omega$, and $Y_n = Y$, for all $n < \omega$. Then the families $(X_n: n < \omega)$ and $(Y_n: n < \omega)$ satisfy the hypothesis of Lemma 4.1, and conclusion c) says that X^ω and Y^ω are countably compact. Clearly $X \cap Y = \omega$ and so $X \times Y$ is not countably compact.

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