
TOPOLOGY PROCEEDINGS



Volume 4, 1979

Pages 577–587

<http://topology.auburn.edu/tp/>

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Topology Proceedings

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ISSN: 0146-4124

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Introduction

It is well known that every countably compact, metacompact (T_1 -) space is compact, and it is easy to see that every pseudocompact, paracompact space is compact. The obvious question, then, is whether every pseudocompact, metacompact space is compact. (Here pseudocompactness is understood to include complete regularity.) Certainly metacompactness cannot be much further weakened: the Mrówka-Isbell space Ψ [GJ, Example 5I] is a locally compact, pseudocompact, (even e -countably compact), orthocompact, and subparacompact--hence θ -refinable--0-dimensional Moore¹ space which is not compact.

An affirmative answer to this question is apparently cited in the otherwise very useful and complete survey paper [Ma]. However, extensive search has failed to turn up the desired result in any of the references therein cited, and the question seems therefore to be still open. The purpose of this note is to show that the result is in fact true.

(I am grateful to C. E. Aull and Jerry Vaughan for bringing this question to my attention.)

Preliminaries

A space X is:

¹I have recently learned that Ψ is literally a Moore space: it is Example 9 of Chapter I of Moore's book [Mo]!

weakly compact (=feebly compact) iff every infinite, disjoint family of non-empty open subsets of X clusters at some point of X ; [equivalently, iff every locally finite family of non-empty open subsets of X is finite];

almost (countably) compact iff every (countable) open cover of X has a finite subfamily whose union is dense in X ;

almost Lindelöf iff every open cover of X has a countable subfamily whose union is dense in X ;

e-countably compact ($e\text{-}\aleph_1\text{-compact}$) iff X has a dense subset, D , every infinite (uncountable) subset of which has a limit point in X ; and

π -regular iff every non-empty open subset of X contains a regularly closed subset of X , i.e., iff $\{cl V : V(\neq \emptyset) \text{ is open in } X\}$ is a π -network for X .

(Almost compact Hausdorff spaces are in some contexts better known as *H-closed* spaces [AU], and π -regularity has also been called quasi-regularity [Ox].

The following facts are well-known, or easy to prove, or both.

Fact 1. A Tikhonov space is pseudocompact iff it is weakly compact, and an e-countably compact space is weakly compact.

Fact 2. A regular, almost compact space is compact.

Fact 3. [FL] A space, X , is a Baire space iff for

every point-finite, open cover, \mathcal{V} , of X , $\{x \in X: \mathcal{V} \text{ is locally finite at } x\}$ is dense (and obviously open) in X .

Fact 4. (Slightly generalizing a result of [McC].)

If X is weakly compact and π -regular, then X is a Baire space. [Sketch of proof: Let $\langle G_n: n \in \omega \rangle$ be a descending sequence of dense, open subsets of X , and let V_0 be a non-empty open subset of X . Recursively choose non-empty open sets V_n ($n \in \omega$) in X such that $\text{cl } V_{n+1} \subseteq V_n \cap G_n$. By the weak compactness of X , $\bigcap_n V_n = \bigcap_n \text{cl } V_n \neq \emptyset$.]

Results

Most of the work is done in the following theorem.

Theorem 1. Let X be a weakly compact, metacompact, Baire space; then X is almost compact.

Proof. Let \mathcal{V} be a point-finite open cover of X , and let $G = \{x \in X: \mathcal{V} \text{ is locally finite at } x\}$; G is dense (and open) in X . For any $A \subseteq X$, let $\text{ST}(A, \mathcal{V}) = \{V \in \mathcal{V}: V \cap A \neq \emptyset\}$, and, as usual, let $\text{st}(A, \mathcal{V}) = \bigcup \text{ST}(A, \mathcal{V})$. Let \mathcal{H} be a family of non-empty open subsets of X such that:

- (1) for each $H \in \mathcal{H}$, $\text{ST}(H, \mathcal{V})$ is finite;
- (2) if $H_0, H_1 \in \mathcal{H}$, and $H_0 \neq H_1$, then $H_0 \cap \text{st}(H_1, \mathcal{V}) = \emptyset$;
and
- (3) \mathcal{H} is maximal with respect to (1) and (2).

By (2), \mathcal{H} is disjoint. If \mathcal{H} were infinite, it would cluster at some $x \in X$. But then if $x \in V \in \mathcal{V}$, $\{H \in \mathcal{H}: V \cap H \neq \emptyset\}$ would be infinite, which contradicts (2). Thus, \mathcal{H} is finite, and so, of course, if $V_0 = \text{ST}(\bigcup \mathcal{H}, \mathcal{V})$. Moreover, $\bigcup V_0 (= \text{st}(\bigcup \mathcal{H}, \mathcal{V}))$ is dense in X . (For if not, pick $x \in G \cap \text{int}(X \setminus \bigcup V_0)$, and

let $W \subseteq \text{int}(X \setminus \bigcup V_0)$ be an open nbhd of x meeting only finitely many members of V ; clearly $\mathcal{H} \cup \{W\}$ satisfies (1) and (2), so \mathcal{H} is not maximal.) Thus, X is almost compact.

Corollary 1. Let X be π -regular, weakly compact, and metacompact; then X is almost compact.

Corollary 2. Let X be regular and weakly compact; then X is compact iff X is metacompact.

Corollary 3. Let X be a pseudocompact Tikhonov space; then X is compact iff X is metacompact.

Corollary 4. Let X be π -regular, weakly compact, and countably metacompact; then X is almost countably compact.

Remarks

As was noted in the Introduction, metacompactness cannot be weakened to θ -refinability + orthocompactness in any of these results. I do not know whether π -regularity is required in Corollary 1.

Question. Can the requirement that X be π -regular be deleted from Corollary 1? Can it be replaced by a requirement that X be, say, semiregular (i.e., have a base of regularly open sets)?

The following example shows, however, that in Corollary 2 'regular' cannot be replaced by 'Hausdorff, semiregular, and π -regular'.

Example 1. Let $Y = \omega^2 + 1$, and let $F = \{\omega \cdot n : n \in \omega\}$. (All arithmetic is ordinal arithmetic.) Let λ be the order

topology on Y , and let τ be the topology on Y generated by $\lambda \cup \{Y \setminus F\}$. Let X be that quotient of the discrete union of two copies of $\langle Y, \tau \rangle$ obtained by identifying the two copies of F in the obvious way. It is easy to verify that X is Hausdorff, π -regular, semiregular, non-regular, weakly compact, metacompact, almost compact (of course), and not compact. (In fact, X is a non-compact, minimal Hausdorff space; see [SS, Example 100].)

In another direction, it is well-known that a pseudo-compact T_4 -space is countably compact. One might therefore wonder whether replacement of ' π -regular' in Corollary 4 by 'Tikhonov' would permit the conclusion to be strengthened to ' X is countably compact'. However, this is not the case: let $X = [(\omega_1 + 1) \times (\omega + 1)] \setminus \{(\omega_1, \omega)\}$, the Tikhonov Plank. Then X is Tikhonov, weakly compact, and countably metacompact, but not countably compact.

Finally, both Example 1 and the Tikhonov Plank refute the conjecture that if X is π -regular and weakly compact, then any point-finite family, \mathcal{U} , of non-empty subsets of X has a finite subfamily whose union is dense in $\bigcup \mathcal{U}$. In Example 1, take \mathcal{U} to be the set of isolated singletons, and in the Tikhonov Plank take \mathcal{U} to be $\{(\omega_1 + 1) \times \{n\} : n \in \omega\}$. In each case \mathcal{U} is even disjoint, and $\bigcup \mathcal{U}$ is dense in the space. Note, however, that the proof of Corollary 1 *does* imply the special case of the conjecture in which \mathcal{U} covers X , which is the result claimed in [Ma].

Another possible extension of Theorem 1 and its corollaries is suggested by the fact that every countably compact,

metaLindelöf T_1 -space is compact. (This result has been folklore for many years.) Thus, we may ask whether meta-compactness can be replaced by metaLindelöfness in any (or all) of the foregoing results (excluding Corollary 4, of course). Under the Continuum Hypothesis (CH) the answer is 'no,' even for Corollary 3.

Example 2. [CH] Let Z be the product of ω copies of the lexicographically ordered Cantor square (i.e., the product of the middle-thirds Cantor set with itself). Then Z is first countable, compact, 0-dimensional, and Hausdorff, $w(Z) = |Z| = 2^\omega$, and, if $A \subseteq Z$ with $|A| < 2^\omega$, then A is nowhere dense in Z . Let $Y = Z \times \omega$ with the usual product topology. Let \mathcal{C} be the family of non-empty, clopen subsets of Z , and, for each $n \in \omega$, let $\mathcal{C}_n = \{C \times \{n\} : C \in \mathcal{C}\}$. Let $K = \bigcup \{\mathcal{C}_n : n \in \omega\}$. Finally, let Σ be the set of all countably infinite subsets, \mathcal{S} , of K such that for each $n \in \omega$, $|\mathcal{S} \cap \mathcal{C}_n| \leq 1$. If $\{\mathcal{S}_0, \mathcal{S}_1\} \in \Sigma$, say that \mathcal{S}_0 and \mathcal{S}_1 are *almost disjoint* iff $\{S_0, S_1 \in \mathcal{S}_0 \times \mathcal{S}_1 : S_0 \cap S_1 \neq \emptyset\}$ is finite. Of course, a family $\Delta \subseteq \Sigma$ is almost disjoint iff \mathcal{S}_0 and \mathcal{S}_1 are almost disjoint for any distinct $\mathcal{S}_0, \mathcal{S}_1 \in \Delta$.

$|\Sigma| = 2^\omega$, so let $\Sigma = \{\mathcal{S}_\alpha : \alpha < 2^\omega\}$ be a 1-1 enumeration. Also, $|Z| = 2^\omega$, so let $Z = \{z_\alpha : \alpha < 2^\omega\}$ be a 1-1 enumeration, and, for $\alpha < 2^\omega$, let $Z_\alpha = \{z_\beta : \beta < \alpha\}$; note that $|Z_\alpha| < 2^\omega$, so that Z_α is nowhere dense in Z . By recursion on $\alpha < 2^\omega$ construct $\mathcal{D}_\alpha \in \Sigma \cup \{\emptyset\}$ and $\Delta_\alpha \subseteq \Sigma \cup \{\emptyset\}$ so that the following hold for each $\alpha < 2^\omega$:

- (i) $\Delta_\alpha = \{\mathcal{D}_\beta : \beta \leq \alpha\}$;
- (ii) Δ_α is almost disjoint;

(iii) there is a $\bar{D} \in \Delta_\alpha$ such that \bar{D} is not almost disjoint from \mathcal{S}_α ;

(iv) $(Z \times \{n\}) \cap \bigcup \bar{D}_\alpha \subseteq (Z \times \{n\}) \cap \bigcup \mathcal{S}_\alpha$ for each $n \in \omega$;
and

(v) $(Z_\alpha \times \omega) \cap \bigcup \bar{D}_\alpha = \emptyset$.

Suppose that $\alpha < 2^\omega$, and that \bar{D}_β and Δ_β have been defined for all $\beta < \alpha$ to satisfy (i)-(v). Let $\Delta'_\alpha = \bigcup \{\Delta_\beta : \beta < \alpha\}$; clearly Δ'_α is almost disjoint. If $\Delta'_\alpha \cup \{\mathcal{S}_\alpha\}$ is not almost disjoint, let $\bar{D}_\alpha = \emptyset$. Otherwise, let $M = \{n \in \omega : \mathcal{S}_\alpha \cap C_n \neq \emptyset\}$, and, for each $n \in M$, let $C_n = (Z \times \{n\}) \cap \bigcup \mathcal{S}_\alpha$. For each $n \in M$ there is a $C'_n \in C_n$ such that $C'_n \subseteq C_n \setminus (Z_\alpha \times \{n\})$; let $\bar{D}_\alpha = \{C'_n : n \in M\}$. In either case set $\Delta_\alpha = \Delta'_\alpha \cup \{\bar{D}_\alpha\}$: (i)-(v) are clearly satisfied at α .

Now let $\Delta = \bigcup \{\Delta_\alpha : \alpha < 2^\omega\} \setminus \{\emptyset\}$; clearly Δ is a maximal almost disjoint subfamily of Σ . Moreover, for any $y \in Y$, $\{\bar{D} \in \Delta : y \in \bigcup \bar{D}\}$ is evidently countable. (It is here that CH is used.)

Let $X = Y \cup \Delta$, and topologize X by letting Y be an open subspace of X and taking as basic open nbhds of a point $\bar{D} \in \Delta$ all sets of the form $\{\bar{D}\} \cup (\bar{D} \setminus \mathcal{J})$, where \mathcal{J} is a finite subset of \bar{D} . Clearly X is *Tikhonov*, *0-dimensional*, *first countable*, and *locally compact*, and Y is a *dense, Lindelöf* (even σ -compact) *subspace of X*. Moreover, X is *metaLindelöf*, since Y is Lindelöf, and the family of basic open nbhds of points of Δ is--by the last observation of the preceding paragraph--point-countable. Clearly X is *not compact*, so it only remains to show that X is *weakly compact*. Let $\mathcal{V} = \{V_n : n \in \omega\}$ be a disjoint family of non-empty, open subsets of X . Y is dense in X , so we may assume that $\bigcup \mathcal{V} \subseteq Y$. If

there is an $m \in \omega$ such that $V_n \cap (Z \times \{m\}) \neq \emptyset$ for infinitely many $n \in \omega$, then clearly the compactness of Z ensures that some point of $Z \times \{m\}$ is a cluster point of V . Otherwise, there is an $\mathcal{S} = \{S_n : n \in \omega\} \in \Sigma$ such that $S_n \subseteq V_n$ for each $n \in \omega$, and the maximality of Δ then ensures that some $\bar{D} \in \Delta$ is not almost disjoint from \mathcal{S} , i.e., that \bar{D} is a cluster point of V .

(The basic idea of Example 2--to 'fatten-up' the space Ψ --is due to A. Berner, who has shown that by exercising greater care in the construction of Δ one can ensure (a) that X is not e -countably compact and in fact (b) that the one-point compactification of X is a Frechet-Uryson space [Be].)

A positive result is still possible, though, if we strengthen weak compactness to e -countable compactness.

Theorem 2. Let X be e -countably compact and metacompact. Then X is almost compact.

Proof. Let D be a dense subset of X , every infinite subset of which has a cluster point in C , and let V be a point-finite open cover of X . Let $A \subseteq D$ be maximal with respect to the following property: if $x, y \in A$, and $x \neq y$, then $y \notin \text{st}(x, V)$. A is easily seen to be a closed, discrete subset of X , so A must be finite. But then $\bigcup \{\text{ST}(x, V) : x \in A\}$ is finite, and its union, $\text{st}(A, V)$, is dense in X .

Theorem 3. Let X be $e\text{-}\aleph_1$ -compact and metaLindelöf. Then X is almost Lindelöf.

Proof. The proof is *mutatis mutandis* the same as that of Theorem 2.

Corollary 5. Let X be $e\text{-}\aleph_1$ -compact, regular, meta-Lindelöf, and ω_1 -open (i.e., countable intersections of open sets are open). Then X is Lindelöf.

Proof. The proof is similar to the proof of the first part of Fact 1.

Example 2 shows that Corollary 5 fails for spaces that are not ω_1 -open, at least under CH.

Addendum. Since writing the first draft of this paper I have learned that Corollary 3 has also been recently proved by W. S. Watson and O. Förster (independently). Watson's proof suggested the following lemma, which may be of interest in its own right.

Lemma 1. For any space X the following are equivalent:

- (a) X is Baire,
- (b) for any point-finite open cover, \mathcal{V} , of X , $\{x \in X: \mathcal{V} \text{ is locally finite at } x\}$ is dense (and open) in X ;
- (c) for any point-finite open cover, \mathcal{V} , of X , there is a π -base, β , for X such that if $V \in \mathcal{V}$, $B \in \beta$, and $V \cap B \neq \emptyset$, then $B \subseteq V$.

Proof. The equivalence of (a) and (b) is due to Fletcher and Lindgren and is Fact 3 above, and that (a) implies (c) is shown by Watson in [Wa]. To see that (c) implies (a), suppose that X is not Baire. Then there are dense, open sets $G_n \subseteq X$ ($n \in \omega$) and a non-empty, open set $V \subseteq X$ such that $V \cap \bigcap \{G_n: n \in \omega\} \neq \emptyset$. For $n \in \omega$ let $W_n = V \cap G_n$, and let $\mathcal{W} = \{X\} \cup \{W_n: n \in \omega\}$. Clearly \mathcal{W} is a point-finite open cover of X ; but if $B \subseteq V$ is open and non-empty, then clearly

$B \notin \cap \{W \in \mathcal{W} : B \cap W \neq \emptyset\} = \cap \mathcal{W} = \emptyset$, so (c) must fail for the cover \mathcal{W} .

In fact it is not difficult to show directly that (b) and (c) are equivalent, and that moreover if \mathcal{V} is a point-finite open cover of any space X (Baire or not), $\{x \in X : \mathcal{V} \text{ is locally finite at } x\}$ is dense in X iff X has such a π -base as is described in (c).

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