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by

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1. Introduction

We assume that all spaces are T_1. Let X be a topological space and let A be a subset of X. The *character* $\chi(A,X)$ *of* A *in* X is $\omega \cdot \alpha$ where $\alpha = \min \{\kappa$: there is a base $\mathcal{U}$ for the neighborhoods of A in X with $|\mathcal{U}| = \kappa\}$. The *pseudocharacter* $\psi(A,X)$ *of* A *in* X is $\omega \cdot \alpha$, where $\alpha = \min \{\kappa$: there exists a collection $\mathcal{U}$ of open sets in X with $|\mathcal{U}| = \kappa$ and $\cap \mathcal{U} = A\}$. Let $\chi_c(X) = \sup\{\chi(F,X)$: F is a closed subset of X$\}$ and $\psi_c(X) = \sup\{\psi(F,X)$: F is a closed subset of X$\}$. We call $\chi_c(X)$ (resp. $\psi_c(X)$) the *closed character* (resp. *closed pseudocharacter*) of X (cf. [1] and [7]).

In §2 we prove several results concerning χ_c and ψ_c, and in §3 we prove the independence in ZFC of the following: "If X is a normal space such that every closed discrete subset of X has Ulam-nonmeasurable power, and if $\chi_c(X) = \omega$, then X is realcompact." (3.2 provides a partial answer to the following question of Blair [2, 4.9 (c)]: Under MA + $\neg$CH, is every perfectly normal space of Ulam-nonmeasurable power realcompact?)

The author is indebted to the referee for correcting an error in the original version of this paper (see 2.9(b)) and for several other helpful suggestions. The author is also grateful to R. L. Blair for many instructive conversations concerning this paper.

2. Closed Character and Closed Pseudocharacter

Let $s(X)$ and $\psi(X)$ denote the spread of X and the pseudocharacter of X. (For definitions, see [8].) The *extent* $e(X)$ of X is $\omega \cdot \alpha$, where α is the smallest cardinal such that every closed discrete subset of X has cardinality $\leq \alpha$ (see [3, 1.7.12] and [7]). Clearly $e(X) \leq s(X)$ and $\psi(X) \leq \psi_c(X) \leq \chi_c(X)$.

Theorem 2.4 below is essentially due to Aull [1, Theorem 10]. We give a more direct proof, based on the following two lemmas:

2.1. *Lemma. If* A *is a set of nonisolated points of* X, *and if the points of* A *can be separated by disjoint open sets, then* $\chi(A,X) > |A|$.

Proof. The proof uses a well-known technique (cf. [3, 1.4.17]): Let $A = \{x_\xi : \xi < \kappa\}$ be a set of nonisolated points of X and let $\{U_\xi : \xi < \kappa\}$ be a disjoint open collection which separates the points of A. Let $\{V_\xi : \xi < \kappa\}$ be any collection of κ open sets such that $A \subset V_\xi$ for all $\xi < \kappa$. For each $\xi < \kappa$, pick $y_\xi \in U_\xi \cap V_\xi$ with $y_\xi \neq x_\xi$ and let $G = \bigcup_{\xi<\kappa} (U_\xi \cap V_\xi - \{y_\xi\})$. Then $A \subset G$, but for all $\xi < \kappa$, $y_\xi \in V_\xi - G$. Thus $\{V_\xi : \xi < \kappa\}$ is not a base for the neighborhoods of A in X.

2.2. *Lemma. If* X *is regular and if* A *is a countably infinite discrete set of nonisolated points of* X, *then* $\chi(A,X) \geq \omega_1$.

Proof. It is easy to see that the points of A can be separated by disjoint open sets (cf. [4, 2.1]). Thus the result follows from 2.1.

2.3. *Example.* Let Z be the unit interval [0,1] topologized by adding to the usual topology sets of the form $[0,\varepsilon) - \{1/n: n\in N\}$ $(0<\varepsilon\leq 1)$. The set $A = \{0\} \cup \{1/n: n\in N\}$ is a countably infinite (closed) discrete subset of the Hausdorff space Z consisting of nonisolated points of Z. It is easy to see that there is a countable neighborhood base $\mathcal{U}$ for A in the usual topology on [0,1], and $\mathcal{U}$ is also a neighborhood base for A in Z; thus "regular" cannot be replaced by "Hausdorff" in 2.2.

2.4. *Theorem (Aull).* *If* X *is a regular space and* $\chi_c(X) = \omega$, *then the set of nonisolated points of* X *is countably compact.*

Proof. Let I be the set of all isolated points of X and let A be a closed discrete subset of X-I. A is a closed discrete subset of X consisting of nonisolated points of X and hence is finite by 2.2.

2.5. *Remark.* The referee notes that if X is strongly Hausdorff [8] and A is as in 2.2, then there is an infinite $B \subset A$ such that $\chi(B,X) > \omega$. It follows that 2.4 holds with "regular" replaced by "strongly Hausdorff."

2.6. *Theorem.* *Let* κ *be an infinite regular cardinal. If every closed discrete subset of* X *has cardinality* $<\kappa$ *and if* $\psi_c(X) < \kappa$, *then every discrete subset of* X *has cardinality* $<\kappa$.

Proof. The proof is like that of [11, Theorem 2]. Let $\psi_c(X) = \alpha$, let D be a discrete subset of X, and let A be the set of all limit points of D in X. A is closed in X and

hence $X - A = \bigcup_{\xi<\alpha} F_\xi$, where each F_ξ is closed in X. Then $|D| = |\cup_{\xi<\alpha} \overline{D} \cap F_\xi| \leq \Sigma_{\xi<\alpha} |\overline{D} \cap F_\xi|$. Since each $\overline{D} \cap F_\xi$ is closed discrete, $|\overline{D} \cap F_\xi| < \kappa$ and thus $|D| < \kappa$.

2.7. *Remark.* Under the hypotheses of 2.6, if κ is weakly compact and X is Hausdorff, then we can conclude that $s(X) < \kappa$ since in Hausdorff spaces weakly compact spreads are attained [8, p. 40].

A Tychonoff space X is *almost compact* if $|\beta X - X| \leq 1$ (see [5, 6J]).

2.8. *Theorem. If* X *is normal and almost compact, then* $\psi(F,X) = \chi(F,X)$ *for every closed subset* F *of* X.

Proof. Since $\psi(F,X) = \chi(F,X)$ for all closed subsets F of a compact space X, we may assume that X is not compact. Then $\beta X = X \cup \{\infty\}$, where $\infty \notin X$ and basic neighborhoods of ∞ are complements in βX of compact subsets of X.

Let F be closed in X. If F is compact, then $cl_{\beta X} F = F$ and the result follows from the fact that $\psi(F,\beta X) = \chi(F,\beta X)$. Therefore we may assume that F is not compact. It suffices to show that for $U \subset X$, U is an open set in X containing F if and only if $U \cup \{\infty\}$ is an open set in βX containing $cl_{\beta X} F$.

Let $F \subset U$ with U open in X. Then F and X-U are disjoint closed subsets of X and thus have disjoint zero-set neighborhoods Z_1 and Z_2, respectively. Since X is almost compact, either Z_1 or Z_2 is compact [5, 6J], and thus Z_2 is compact since F is not. Hence X-U is compact, which implies that $U \cup \{\infty\}$ is an open neighborhood of $F \cup \{\infty\} = cl_{\beta X} F$.

Conversely, if $U \cup \{\infty\}$ is an open set in βX containing

$cl_{\beta X}F$, then X-U is compact, and therefore U is an open set in X containing F.

2.9. *Remarks.* (a) The referee has supplied the following example which shows that the hypothesis of normality cannot be omitted in 2.8: Let $S = [0,\omega_1]^2$, $X = S - \{\langle\omega_1,\omega_1\rangle\}$, and $F = [0,\omega_1) \times \{\omega_1\}$. It is well known that $\beta X = S$ [5, 8L]; F is closed in X and $\psi(F,X) = \omega_1$, but $\chi(F,X) > \omega_1$.

(b) In [6], Ginsburg asserts that (*) if X is Hausdorff, then $\chi(\Delta,X\times X) = \omega$ if and only if X is metrizable and the set of nonisolated points of X is compact. The proof of (*) in [6] is in error (in the assertion that any countably infinite closed discrete set of nonisolated points of a first countable space Y has uncountable character in Y; we need only note that if A is as in 2.3, then $\chi(A,Z) = \omega$), and in an earlier version of this paper we claimed that regularity of X was needed for (*). We are grateful to the referee for pointing out that (*) is, in fact, correct as stated, and that its proof can be corrected by the following:

2.10. *Proposition. If* $\chi(\Delta,X\times X) = \omega$, *then the set of nonisolated points of* X *is countably compact.*

Proof. Let A be an infinite closed discrete set of nonisolated points of X. Let $\Delta_A = \{\langle x,x\rangle: x \in A\}$. Then $\chi(\Delta_A,A\times X) \leq \chi(\Delta,X\times X)$, but since the points of Δ_A can be separated by disjoint open sets in $A\times X$, $\chi(\Delta_A,A\times X) > \omega$ by 2.1.

3. Some Independence Results

We shall show the independence in ZFC of each of the

following:

P. If X is normal and $\chi_c(X) = \omega$, then X is paracompact.

Q. If X is normal, $\chi_c(X) = \omega$, and every closed discrete subset of X is of Ulam-nonmeasurable power, then X is realcompact.

3.1. *Theorem* [MA + ¬CH]. *Every regular space with countable closed character is paracompact.*

Proof. Let I be the set of all isolated points of X. By 2.4, X-I is countably compact. Hence, by Weiss's theorem [12, Corollary 3], X-I is compact. Thus X is the union of a compact space and a set of isolated points and is therefore paracompact.

3.2. *Theorem* [MA + ¬CH]. *If X is a regular space with countable closed character, and if every closed discrete subset of X is of Ulam-nonmeasurable power, then X is realcompact.*

Proof. Let I be the set of all isolated points of X. |I| is Ulam-nonmeasurable by 2.6, and thus I is realcompact [5, 12.2]. As in the proof of 3.1, X-I is compact, and thus X is realcompact [5, 8.16].

3.3. *Remarks.* (a) Alternatively, 3.2 follows from 3.1 and Katětov's theorem [9, Theorem 3] (which asserts that paracompact spaces without closed discrete subsets of Ulam-measurable power are realcompact).

(b) Under the hypothesis of 3.2 |X| is Ulam-nonmeasurable. (Since Ulam-measurable cardinals are weakly compact

[8, A6.5, and A6.3], this follows from 2.7 and the fact that $|X| \leq \exp \exp s(X)$ [8, 2.9].)

3.4. *Theorem* [◇]. *There exists a normal space with countable closed character and countable extent which is neither paracompact nor realcompact.*

Proof. Ostaszewski's space Ω_0 [10] is perfectly normal, countably compact, almost compact, but not compact. Thus $e(\Omega_0) = \omega$, and by 2.8, $\chi_c(\Omega_0) = \omega$, but Ω_0 is neither paracompact nor realcompact.

3.5. *Remarks.* (a) The fact that $\chi_c(\Omega_0) = \omega$ also follows from Aull's theorem [1, Theorem 5] that every perfectly normal countably compact space has countable closed character.

(b) 3.1 and 3.2 establish the consistency of P and Q, while 3.4 establishes the consistency of ¬P and ¬Q. (This simultaneous use of Ostasewski's example and Weiss's theorem is not unlike [2, 4.9].)

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