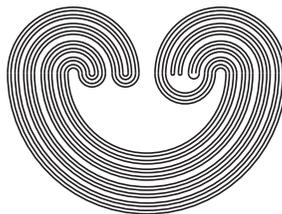


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## Research Announcement:

### THE SPAN OF MAPPINGS AND SPACES

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## THE SPAN OF MAPPINGS AND SPACES

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Let  $X, Y$  be metric spaces, and let  $f: X \rightarrow Y$  be a mapping. By  $p_1$  and  $p_2$  we denote the standard projections of the product  $X \times X$  onto  $X$ , i.e.,  $p_1(x, x') = x$  and  $p_2(x, x') = x'$  for  $(x, x') \in X \times X$ . The *span*  $\sigma(f)$  of the mapping  $f$  is the least upper bound of the set of real numbers  $\alpha$  with the following property: there exist connected sets  $C_\alpha \subset X \times X$  such that  $p_1(C_\alpha) = p_2(C_\alpha)$  and  $\alpha \leq \text{dist}[f(x), f(x')]$  for  $(x, x') \in C_\alpha$  (see [2], p. 99). The span  $\sigma(X)$  of the space  $X$  is the span of the identity mapping on  $X$  (see [4], p. 209). The purpose of the present paper<sup>(1)</sup> is to announce some results which relate to spans of mappings and have a number of interesting consequences for spans of spaces. A complete version will be published elsewhere.

The proofs of the following four propositions are rather straightforward.

1. If  $f: X \rightarrow Y$ , then  $0 \leq \sigma(f) \leq \sigma(Y) \leq \text{diam } Y$ .
2. If  $f: X \rightarrow Y$  and  $X$  is compact, then  
 $\text{Inf}\{d[f^{-1}(y), f^{-1}(y')]: \sigma(f) \leq \text{dist}(y, y')\} \leq \sigma(X)$ .
3. If  $f: X \rightarrow Y$ ,  $X$  is compact and  $0 < \epsilon \leq \text{diam } Y$ , then  
 $0 < \text{Inf}\{d[f^{-1}(y), f^{-1}(y')]: \epsilon \leq \text{dist}(y, y')\}$ .
4. If  $f: X \rightarrow Y$  and  $X$  is compact, then  $\sigma(X) = 0$  implies  $\sigma(f) = 0$ .

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<sup>1</sup>This paper was presented during the Thirteenth Spring Topology Conference at Ohio University, on March 17, 1979.

Note that proposition 4 follows from propositions 2 and 3. By  $S$  we denote the unit circle on the plane, and by  $T$  we denote the union of two tangent circles each of radius  $1/2\pi$ . We consider  $T$  to be a metric space with the geodesic metric  $\rho$ . In other words,  $\rho(y, y')$  is the length of the shortest arc joining the points  $y$  and  $y'$  in  $T$  for  $y, y' \in T$ , so that the diameter of  $T$  is one. We say that a mapping is *essential* if it is not homotopic to a constant mapping.

5. *Lemma.* If  $f: S \rightarrow T$  is an essential mapping and  $0 \leq \epsilon \leq \frac{1}{2}$ , then there exist a continuum  $K$  and two surjective mappings  $\phi, \psi: K \rightarrow S$  such that

$$\rho[f\phi(x), f\psi(x)] = \epsilon \quad (x \in K).$$

6. *Theorem.* If  $f: X \rightarrow T$  is an essential mapping,  $X$  is compact,  $\dim X \leq 1$  and  $0 \leq \epsilon \leq \frac{1}{2}$ , then there exists a continuum  $K \subset X \times X$  such that  $p_1(K) = p_2(K)$  and  $\rho[f(x), f(x')] = \epsilon$  for  $(x, x') \in K$ .

The following four statements are corollaries to theorem 6.

7. If  $f: X \rightarrow T$  is an essential mapping,  $X$  is compact and  $\dim X \leq 1$ , then  $\sigma(f) \geq \frac{1}{2}$ .

8. If  $f: X \rightarrow T$  is an essential mapping,  $X$  is compact,  $\dim X \leq 1$  and  $0 < \epsilon \leq \frac{1}{2}$ , then

$$0 < \inf\{d[f^{-1}(y), f^{-1}(y')]: \rho(y, y') = \epsilon\} \leq \sigma(X).$$

9. If  $X$  is compact and  $\sigma(X) = 0$ , then each mapping  $f: X \rightarrow T$  is inessential.

10. If  $X$  is a continuum and  $\sigma(X) = 0$ , then  $X$  is tree-like.

It is known [4] that continua of span zero are one-dimensional if non-degenerate. By corollary 9, the mappings defined on them and having values in one-dimensional polyhedra [3] are all inessential, and then corollary 10<sup>(2)</sup> can be obtained via a well-known characterization of tree-like continua [1]. Also, notice that  $\sigma(T) = \frac{1}{2}$ . Hence, by proposition 1 and corollary 7, we get  $\sigma(f) = \sigma(T)$  for all essential mappings  $f$  of one-dimensional compact metric spaces into  $T$ . It remains as an open problem to determine a wider class of mappings  $f: X \rightarrow Y$  such that  $\sigma(f) = \sigma(Y)$ .

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<sup>2</sup>A recent result of James F. Davis establishes the equality between the span and the semi-span [5] for a certain class of continua. Using the tree-likeness of continua of span zero (corollary 10), it implies, among other things, that they have the fixed point property.