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Research Announcement:

THE SPAN OF MAPPINGS AND SPACES

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THE SPAN OF MAPPINGS AND SPACES

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Let X, Y be metric spaces, and let $f: X \to Y$ be a mapping. By p_1 and p_2 we denote the standard projections of the product $X \times X$ onto X, i.e., $p_1(x,x') = x$ and $p_2(x,x') = x'$ for $(x,x') \in X \times X$. The *span* $\sigma(f)$ of the mapping f is the least upper bound of the set of real numbers α with the following property: there exist connected sets $C_\alpha \subset X \times X$ such that $p_1(C_\alpha) = p_2(C_\alpha)$ and $\alpha \leq \text{dist}[f(x), f(x')]$ for $(x,x') \in C_\alpha$ (see [2], p. 99). The span $\sigma(X)$ of the space X is the span of the identity mapping on X (see [4], p. 209). The purpose of the present paper[1] is to announce some results which relate to spans of mappings and have a number of interesting consequences for spans of spaces. A complete version will be published elsewhere.

The proofs of the following four propositions are rather straightforward.

1. *If* $f: X \to Y$, *then* $0 \leq \sigma(f) \leq \sigma(Y) \leq \text{diam } Y$.

2. *If* $f: X \to Y$ *and* X *is compact, then*
$$\text{Inf}\{d[f^{-1}(y), f^{-1}(y')]: \sigma(f) \leq \text{dist}(y,y')\} \leq \sigma(X).$$

3. *If* $f: X \to Y$, X *is compact and* $0 < \varepsilon \leq \text{diam } Y$, *then*
$$0 < \text{Inf}\{d[f^{-1}(y), f^{-1}(y')]: \varepsilon \leq \text{dist}(y,y')\}.$$

4. *If* $f: X \to Y$ *and* X *is compact, then* $\sigma(X) = 0$ *implies* $\sigma(f) = 0$.

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[1]This paper was presented during the Thirteenth Spring Topology Conference at Ohio University, on March 17, 1979.

Note that proposition 4 follows from propositions 2 and 3. By S we denote the unit circle on the plane, and by T we denote the union of two tangent circles each of radius $1/2\pi$. We consider T to be a metric space with the geodesic metric ρ. In other words, $\rho(y,y')$ is the length of the shortest arc joining the points y and y' in T for $y,y' \in T$, so that the diameter of T is one. We say that a mapping is *essential* if it is not homotopic to a constant mapping.

5. *Lemma. If* $f: S \to T$ *is an essential mapping and* $0 \leq \varepsilon \leq \frac{1}{2}$, *then there exist a continuum* K *and two surjective mappings* $\phi,\psi: K \to S$ *such that*

$$\rho[f\phi(x), f\psi(x)] = \varepsilon \qquad (x \in K).$$

6. *Theorem. If* $f: X \to T$ *is an essential mapping,* X *is compact,* $\dim X \leq 1$ *and* $0 \leq \varepsilon \leq \frac{1}{2}$, *then there exists a continuum* $K \subset X \times X$ *such that* $p_1(K) = p_2(K)$ *and* $\rho[f(x), f(x')] = \varepsilon$ *for* $(x,x') \in K$.

The following four statements are corollaries to theorem 6.

7. *If* $f: X \to T$ *is an essential mapping,* X *is compact and* $\dim X \leq 1$, *then* $\sigma(f) \geq \frac{1}{2}$.

8. *If* $f: X \to T$ *is an essential mapping,* X *is compact,* $\dim X \leq 1$ *and* $0 < \varepsilon \leq \frac{1}{2}$, *then*

$$0 < \mathrm{Inf}\{d[f^{-1}(y), f^{-1}(y')]: \rho(y,y') = \varepsilon\} \leq \sigma(X).$$

9. *If* X *is compact and* $\sigma(X) = 0$, *then each mapping* $f: X \to T$ *is inessential.*

10. *If* X *is a continuum and* $\sigma(X) = 0$, *then* X *is tree-like.*

It is known [4] that continua of span zero are one-dimensional if non-degenerate. By corollary 9, the mappings defined on them and having values in one-dimensional polyhedra [3] are all inessential, and then corollary 10[2] can be obtained via a well-known characterization of tree-like continua [1]. Also, notice that $\sigma(T) = \frac{1}{2}$. Hence, by proposition 1 and corollary 7, we get $\sigma(f) = \sigma(T)$ for all essential mappings f of one-dimensional compact metric spaces into T. It remains as an open problem to determine a wider class of mappings $f: X \to Y$ such that $\sigma(f) = \sigma(Y)$.

References

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[2] A recent result of James F. Davis establishes the equality between the span and the semi-span [5] for a certain class of continua. Using the tree-likeness of continua of span zero (corollary 10), it implies, among other things, that they have the fixed point property.