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A NOTE ON INFINITE SEQUENTIAL ORDER

by

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I. Introduction

In this paper the example space K_{ω_0} of [1] is proven, in Theorem 2.2, to be the test space for regular sequential spaces of order ω_0. A *test space for sequential spaces of order* α, where α is an ordinal $\leq \omega_1$, is a space T_α such that if X is any sequential space of order α, then there is a subspace Y of X such that the sequential extension of Y is homeomorphic to T_α. The *sequential extension* [1] of a space X is denoted $s(X)$ and is defined to be the set X retopologized by letting all of the sequentially open sets be open. The Lemma 2.1, which was established to prove this theorem, provides a procedure for selecting a sequence of subspaces T_i such that $T_i \subset X - \cup \{T_k : k < i\}$, $\sigma(T_i) \geq i$ and $\sigma(X - \cup \{T_k : k \leq i\}) = \omega_0$. This lemma appears to be a promising approach to prove test space theorems for all regular sequential spaces whose sequential order is a limit ordinal. The rather fascinating fact discovered here is that this lemma is false for all ordinals $\beta \geq \omega_0 + 1$, which are the sum of two smaller ordinals, as Example 2.3 shows. These examples show again the unpredictable behavior of the ordinal invariant σ, even on closed or open subspaces. The question of whether the spaces K_β, $\beta < \omega_1$ are the test spaces for regular sequential spaces whose order is greater than

ω_0 is answered negatively here by a transfinite extension of the example of V. Kannan [4, Ex. 1.2, page 197]. Kannan has provided a retopologized variation of K_{ω_0+1}, denoted B_1, which has sequential order ω_0+1 and does not contain a copy of K_{ω_0+1}. Also, K_{ω_0+1} does not contain a copy of B_1. Hence neither K_{ω_0+1} nor B_1 can be test spaces for spaces of sequential order ω_0+1. Transfinite extensions of Kannan's example are constructed in Example 2.5 providing spaces of order α for all $\omega_0 < \alpha < \omega_1$ for which the spaces K_α are not test spaces in the sense of sequential extensions. Hence this brief paper presents the sharpest possible theorem related to test spaces for sequential spaces of infinite order, in the sense that $\sigma(X) \geq \beta$ implies the existence of a subspace V_β whose sequential extension is a space of a given type. It is a continuation of studies from [1], [2], [3], and [4] and the reader is referred to [1] or [2] for the necessary definitions and notation.

II. Results

Complement Lemma 2.1. *Let* $\sigma(X) = \alpha$ *where* α *is not the sum of two ordinals strictly less than* α. *If* A *is an open or closed subspace with* $\sigma(A) = \beta < \alpha$, *then* $\sigma(X - A) = \alpha$.

Proof. Let A be closed. Assume $\sigma(X - A) = \gamma < \alpha$. Since $\sigma(X) = \alpha$, there exists a set $B \subset X$ such that $B^\mu \neq \mathrm{cl}_X(B)$ for $\mu > \gamma + 1 + \beta$. Let $p \in B^{\mu+1} - B^\mu$. Let $C = B - A$ and $T = X - A$. Suppose $p \in T$. Since $C^\gamma_T \subset B^\mu$

and $p \notin B^{\mu}$, $p \notin C_T^{\gamma}$. Since C_T^{γ} is closed in T, there exists a neighborhood U of p such that $U \subset T$ and $U \cap C_T^{\gamma} = \emptyset$. Since T is open $U \cap C_T^{\gamma} = U \cap (C^{\gamma} \cap T) = U \cap C^{\gamma} = \emptyset$. Thus $U \cap (B - A) = \emptyset$ and $U \cap (B \cap A) = \emptyset$. Hence we have the contradiction $U \cap B = \emptyset$. Accordingly, $p \in A$. Since $(B \cap A)^{\beta}$ is closed in A and thus closed in X and $(B \cap A)^{\beta} \subset B^{\mu}$, $p \notin (B \cap A)^{\beta} = \mathrm{cl}_X(B \cap A)$. Thus there exists a neighborhood U of p such that $U \cap (B \cap A) = \emptyset$. Since $p \in B^{\mu+1} - B^{\mu}$, $U \cap B^{\mu} \neq \emptyset$. Since $p \notin \mathrm{cl}_X(B \cap A)$, $p \in \mathrm{cl}_X(B - A) = \mathrm{cl}_X(C)$. Say $p \in C^{\nu}$. Since $C^{\gamma} \subset B^{\gamma} \subset B^{\mu}$, $p \notin C^{\gamma}$ and $\nu > \gamma$. Since $\sigma(X - A) = \gamma$, $(C^{\gamma+1} - C^{\gamma}) \cap T = \emptyset$. Thus $C^{\gamma+1} - C^{\gamma} \subset A$ and since A is closed $C^{\delta+1} - C^{\delta} \subset A$ for each $\delta \geq \gamma$. Since $p \in C^{\nu}$ for some $\nu > \gamma$ and $p \notin C^{\gamma}$, $p \in \mathrm{cl}_X(C^{\gamma+1} - C^{\gamma})$. Since $C^{\gamma+1} - C^{\gamma} \subset A$ and A is closed and $\sigma(A) = \beta$, $p \in \mathrm{cl}_X(C^{\gamma+1} - C^{\gamma}) = (C^{\gamma+1} - C^{\gamma})^{\beta} \subset C^{\gamma+1+\beta}$. However, $C^{\gamma+1+\beta} \subset B^{\gamma+1+\beta} \subset B^{\mu}$, because $\mu > \gamma + 1 + \beta$ and this implies the contradiction, $p \in B^{\mu}$. Thus $\sigma(X - A) = \alpha$ and this completes the proof for the case where A is a closed set. Suppose A is an open set. Then X - A is closed and the assumption that $\sigma(X - A) = \gamma < \alpha$ results in the same contradiction from the preceding proof. This completes the proof.

The preceding (complement) lemma was established for the case $\alpha = \omega_o$, to prove Theorem 2.2. The extension to all ordinals which are not the sum of smaller ordinals is the result of an inquiry by the referee. This lemma provides a means of successively selecting a closed or open subspace from the remaining space whose sequential order is sufficiently large.

The existence of the open subspaces of desired order is a consequence of proof of Proposition 3.1 in [1] and is established as follows. For any sequential space Y where $\sigma(Y) = \omega_o$, for each $n < \omega_o$ there exists a subspace Y' selected from the sequential closures of a set A such that $s(Y') = S_n$. Thus if y is the base point of Y' and p is any fixed point in the sequence in Y' converging to y there are disjoint open sets H_{n-1} and H_n such that $p \in H_{n-1}$ and $y \in H_n$. Hence $A \cap H_{n-1}$ and $A \cap H_n$ are subsets of H_{n-1} and H_n respectively which require at least n-1 and n sequential closures in H_{n-1} and H_n and thus $\sigma(H_{n-1}) \geq n-1$ and $\sigma(H_n) \geq n$. Thus, if $\sigma(Y) = \omega_o$, for each $n < \omega_o$ there are disjoint open subspaces H_{n-1} and H_n such that $\sigma(H_{n-1}) \geq n-1$ and $\sigma(H_n) \geq n$.

Theorem 2.2. *If* X *is a regular sequential space and* $\sigma(X) = \omega_o$, *then* X *contains a subspace* T *such that* $s(T) = K_{\omega_o}$.

Proof. Let $\sigma(X) = \omega_o$. There exist open subspaces V_1 and U_2 such that $V_1 \cap U_2 = \emptyset$, $\sigma(V_1) \geq 1$ and $\sigma(U_2) > 1$ and thus $\sigma(X - V_1) > 1$. If $\sigma(V_1) = \omega_o$, then let T_1 be an S_1 in U_2 and let $G_1 = U_2$. If $\sigma(V_1) = m < \omega_o$, by the complement lemma, $\sigma(X - V_1) = \omega_o$. Then let T_1 be an S_1 in V_1 and let $G_1 = V_1$. Let $X_1 = X - G_1$. Assume this process has been repeated n-1 times. That is, for each $k \leq n-1$, $X_k = X_{k-1} - G_k$, where G_k is open in X_{k-1}, T_k is a subspace of G_k such that $s(T_k) = S_k$ and $\sigma(X_k) = \omega_o$. Since

$\sigma(X_{n-1}) = \omega_o$, there exists open subspaces V_n and U_{n+1} of X_{n-1} such that $V_n \cap U_{n+1} = \emptyset$, $\sigma(V_n) \geq n$, $\sigma(U_{n+1}) > n$ and thus $\sigma(X_{n-1} - V_n) > n$. If $\sigma(V_n) = \omega_o$, then let T_n be a subspace of U_{n+1} such that $s(T_n) = S_n$ and let $G_n = U_{n+1}$. If $\sigma(V_n) = m < \omega_o$, then let T_n be a subspace of V_n such that $s(T_n) = S_n$ and let $G_n = V_n$. By the complement lemma, $\sigma(X_{n-1} - V_n) = \omega_o$. Let $X_n = X_{n-1} - G_n$. Thus, G_n is an open subspace of X_{n-1}, T_n is a subspace of G_n such that $s(T_n) = S_n$ and $\sigma(X_n) = \omega_o$. This completes the induction step and for each $n < \omega_o$, T_n is a subspace of G_n, such that $s(T_n) = S_n$. For each $n < \omega_o$, let p_n be the base point of T_n and let $H = \{p_n: n < \omega_o\}$. If H has no cluster point, then since X is regular there is a disjoint collection of open subsets of X, $\{C_n: n < \omega_o\}$ such that $p_n \in C_n$, for each $n < \omega_o$. If H has cluster point there is a subsequence of $\{p_n\}$ that converges to some point. Thus there is in this case a disjoint collection of open sets in X each containing exactly one point of the convergent subsequence. Hence in either case, there is a sequence of base points $\{p_k\}$ and a disjoint collection of open sets U_k such that $p_k \in U_k$, for $k < \omega_o$. Let $T'_k = U_k \cap T_k$ for each k. Then since the sequential extension of T'_n, $s(T'_n)$, is S_n for each n, $s(\cup\{T'_n: n < \omega_o\}) = \cup\{s(T'_n): n < \omega_o\} = K_{\omega_o}$. Accordingly, $T = \cup\{T'_n: n < \omega_o\}$ is a subspace of X such that $s(T) = K_{\omega_o}$. This completes the proof.

The following examples establish the sharpness of the results in Theorems 2.1 and 2.2. That is, the Complement

Lemma 2.1 is false for all infinite ordinals which are the sum of smaller ordinals and for each ordinal α, $\omega_0 < \alpha < \omega_1$, there is a regular sequential space of order α for which K_α is not a test space using sequential extensions.

Example 2.3. For every ordinal α, $\omega_0 < \alpha < \omega_1$, which is the sum of two ordinals β and γ, $\alpha = \beta + \gamma$, where $\beta < \alpha$ and $\omega_0 \leq \gamma < \alpha$, there is a regular sequential space X, such that $\sigma(X) = \alpha$, which has an open subspace A such that $\sigma(A) = \beta + 1 < \alpha$ and $\sigma(X - A) = \gamma < \alpha$.

For each isolated point, y, of K_γ, let $K_{\beta+1}(y)$ be a copy of $K_{\beta+1}$ with base point 0_y. Form the quotient space X by attaching the base point 0_y of $K_{\beta+1}(y)$ to the isolated point $y \in K_\gamma$. Then $\sigma(X) = \alpha$, $A = \cup\{K_{\beta+1}(y): y \text{ is isolated in } K_\gamma\}$ is an open subspace of X with $\sigma(A) = \beta + 1 < \alpha$ and $\sigma(X - A) = \gamma < \alpha$.

Example 2.4. There is a sequential space T where $\sigma(T) = \omega_0+1$ and an open subspace A of T such that $\sigma(A) < \omega_0+1$ and $\sigma(T - A) < \omega_0+1$.

Let T be the space K_{ω_0+1} and let A be the sequence in T converging to the base point of K_{ω_0+1}. Then $\sigma(A) = 1 < \omega_0+1$ and since T - A is the disjoint union of the spaces S_n, for $n < \omega_0$, $\sigma(T - A) = \omega_0 < \omega_0+1$.

The following example supplies, in two ways, examples of spaces of order α for which K_α is not a test space, under sequential extensions, for all ordinals α such that $\omega_0 < \alpha < \omega_1$.

Example 2.5. For each α, $\omega_0 < \alpha < \omega_1$, there is a sequential space X_α such that $\sigma(X_\alpha) = \alpha$ which does not contain a subspace whose sequential extension is K_α.

The construction of the spaces X_α is by induction on the non-limit ordinals between ω_0 and ω_1. The sequential order at a point p in a space X is defined as $\sigma(p,X) = \inf\{\alpha\colon p \in B^\alpha$, for all $B \subset X$ with $p \in \mathrm{cl}_X(B)\}$. For each ordinal η let $V_\eta = \{x \in X\colon \sigma(x,X) \geq \eta\}$. For $\alpha = \omega_0+1$ let X_{ω_0+1} be the space K_{ω_0+1} retopologized only at the base point 0 in the following way. A neighborhood of 0 is a set $V \subset K_{\omega_0+1}$ such that $0 \in V$, there exists $n < \omega_0$ such that $V_n \subset V$ and $V - \{0\}$ is open as a subset of K_{ω_0+1}. Then X_{ω_0+1} is a sequential space, $\sigma(X_{\omega_0+1}) = \omega_0+1$ and neither X_{ω_0+1} nor K_{ω_0+1} can be embedded in the other, because of the neighborhoods of 0. Suppose X_α has been defined for all non-limit ordinals $\alpha < \beta = \gamma + 1$. In the case where γ is a limit ordinal, choose an increasing sequence of non-limit ordinals $\beta_i \to \gamma$. Form the space X_β by attaching the base point 0_i of X_{β_i} to $\frac{1}{i}$ in S_1 for each i. Let a nhood of the base point 0 (from S_1) be a set V such that $0 \in V$, there exists some $\alpha < \beta$ such that $V_\alpha \subset V$ and $V - \{0\}$ is open in the space $X_\beta - \{0\}$. (This is the disjoint topological sum of the spaces V_{β_i}, $i < \omega_0$.) In the case where γ is a non-limit ordinal, for each $i < \omega_0$ let $X_\gamma(i)$ be a copy of X_γ with base point 0_i. Form the quotient space X_β by attaching the base point 0_i of $X_\gamma(i)$ to

$\frac{1}{i}$ in S_1 for each i. Then in either case X_β is a sequential space, $\sigma(X_\beta) = \beta$. (X_β is K_β retopologized at each point of infinite order.) Neither X_β nor K_β can be embedded in the other because of the neighborhoods of the points of infinite order. For the countable limit ordinals β, let X_β be the disjoint topological sum of the spaces X_α, $\alpha < \beta$. This completes the construction of the spaces X_α, $\omega_o < \alpha < \omega_1$. These examples are rather extreme in the sense that the topology is drastically altered at every point of infinite order in X_α. Another way of building a collection of spaces for which the K_α spaces do not suffice as test spaces can be described as follows. Let $Y_{\omega_o+1} = X_{\omega_o+1}$ from before. Let α be any non-limit ordinal, $\omega_o+1 < \alpha < \omega_1$. Let Y_α be the space K_α with the neighborhoods of only the points of order ω_o+1 altered to have a neighborhood base as in Y_{ω_o+1}. Since any neighborhood of the base point in Y_α must contain infinitely many of the points of order ω_o+1, for the reasons stated before the K_α spaces can not be the test spaces for the spaces Y_α either.

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