TOPOLOGY PROCEEDINGS

Volume 5, 1980

Pages 71–76

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

WEAKLY FLAT CURVES

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At the Oklahoma State University Topology conference in October (78), Daverman asked some questions concerning weakly flat curves. The following theorem gives an answer to one of those questions.

Theorem. Suppose s *is a simple closed curve in* E^3 *which bounds a cellular disk* D_s *which is locally flat mod* s, *and further that each point of* s *is a piercing point for* D_s. *Then* s *is weakly flat.*

Proof. We will prove this by applying the Semicullularity Theorem ([5]) to show that s is toroidal. The defining sequence of tori will clearly be concentric and unknotted ([4]) and this will be sufficient to prove the theorem. The first step is to fatten D_s into a 3-cell K with D_s as the equatorial plane and with Bd K locally flat mod s. K is then considered to be the union of two 3-cells K_1 and K_2 which intersect only in D_s. (All of this is possible since D_s is locally flat at interior points.) Squeeze D_s ([2]) to an arc $\alpha \equiv [0,1]$. Let $K_i' = K_i/D_s \approx \alpha$, i = 1,2. Note that by the Semicullarity Theorem (S.C.T.), each point of Bd K_1' and Bd K_2' is a piercing point. Therefore, we may choose tame arcs b_1, b_1' and b_2, b_2' on Bd K_1' and Bd K_2', respectively, with the following properties:

(1) $b_1 \cap \alpha = 0 = b_2 \cap \alpha$ and $b_1' \cap \alpha = 1 = b_2' \cap \alpha$.

(2) $b_i \cup b_1'$ is a tame arc on K_i', i = 1,2.

Since the complement does not change when D_s is squeezed to α, it is clear that α is cellular and that we could easily obtain an annulus A (or even a sequence of annuli) with the boundary components a_1 and a_2 in Bd K_1' and Bd K_2', respectively, if we could only find a defining sequence with boundary spheres $\{\mathcal{S}_i\}$ for α, and hence D_s, which hit b_1, b_1', b_2 and b_2' in one point each. The $\mathcal{S}_i$ are to be constructed by the same techniques used in the proof of the S.C.T. and such that the A_i's will have the following properties:

(1) The A_i's are disjoint.

(2) Except for some subdisks which may be altered by cutting an pasting, $A_i \subset \mathcal{S}_i$, $\forall i$

(3) If T is the solid torus whose boundary is composed of A and the annulus on K which has a_1 and a_2 for its boundary, then the intersection of the T's is s.

(4) Int $A_i \subset$ Ext K.

We now provide the essential details for obtaining the $\mathcal{S}_i$'s.

There exists a defining sequence $\{\Sigma_n\}$ for α with the property that each sphere in the sequence intersects b_1 and b_1' in one point each (since by S.C.T., the image of $b_1 \cup b_1'$ is tame in E^3/α). Furthermore, the Σ_n may be chosen to intersect Bd K_1' and Bd K_2' only in circles which link α/α, i.e., the "small" disks on Bd$(K_1' \cup K_2')$ which are bounded by these circles contain α/α in their interiors. Also, we shall show that $b_1' \cup b_2'$ is tame so that it has a penetration index ([1]) of two at 1, and thus, there is a defining sequence of spheres for 1 with the property that each sphere intersects b_1' and b_2' in one point each. To see that $b_1' \cup b_2'$ is tame we need only to consider the simple closed curve $b = b_1 \cup b_2 \cup b_1' \cup b_2'$ on $K' \equiv K/D_s \approx \alpha$. b bounds a disk D_b on K' which is locally tame everywhere except on α. We may approximate Int D_b so that b becomes the boundary of a 2-cell which is locally tame mod b or in fact, locally tame mod $b_1 \cup b_1'$. Thus by Theorem 1 of [3], we may conclude that b, and hence, $b_1' \cup b_2'$ is tame.

To complete the argument we choose a defining sequence S_i for 1 with the following properties (see [5], page 11):

(1) Each S_i is the union of two tame disks D_0^i and D_1^i.

(2) Int $D_0^i \subset$ Int K_1' and $D_0^i \cap \alpha$ is a single point d_i.

(3) $D_1^i \cap$ Bd K_1' is the union of a null sequence of tame simple closed curves $\{C_j^i\}_{j=1}^{\infty}$ and a totally disconnected set C^i.

(4) All circles of intersection in $D_1^i \cap$ Bd K_1' hit α (for each i) and $C^i \subset \alpha$.

(5) The S_i's are pairwise disjoint and only one component of $\alpha - S_i - S_j$, $i \neq j$, hits both spheres.

(6) Each S_i hits $b_1' \cup b_2'$ in exactly two points.

Order the S_i in the obvious manner (from larger to smaller) and assume that S_1 is small enough that $0 \in$ Ext S_1. Let q_1 and q_1' denote the points in $\alpha \cap S_1$ with property that $[q_1,1] \cap S_1 = q_1$ and $[0,q_1'] \cap S_1 = q_1'$. Then a new sequence is chosen by choosing S_2 so that it hits α inside $(q_1,1)$ only, i.e. $[0,q_1] \subset$ Ext S_2. By induction, we choose q_1, q_1' for all i.

Let $\varepsilon > 0$ be given and much less than $\text{diam}(\alpha)$. We will construct a 3-cell which contains α and lies within ε of α and whose boundary intersects $b_1 \cup b_2$ and $b_1' \cup b_2'$ twice each. To do this, let a be a disk on Bd K_1' which lies within $N_\varepsilon(\alpha)$ and contains α in its interior. Choose yet another subsequence if necessary so that S_1 is an ε-sphere within $N_\varepsilon(\alpha)$ and $S_1 \cap \text{Bd } K_1' \subset a$. Consider S_i, $i = 1,\cdots,6$. Choose $q \in (q_6,1)$ and $\{\Sigma^j\}$, a defining sequence for $[0,q]$. Notice that if we collapse $[0,q]$ to a point $b_1 \cup b_2$ remains a tame arc by the same proof we first used to show that b and $b_1' \cup b_2'$ are tame. (Our proof will show that $b_1 \cup b_2$ is also tame if we collapse α to a point. However, this does not follow from the previous argument since $D_{b/\alpha}$ is not a disk.) Thus we may choose the defining sequence $\{\Sigma^j\}$ so that each element hits $b_1 \cup b_2$ exactly twice. Then Σ^1 is selected small enough for the following conditions to hold.

(1) $\Sigma^1 \subset N_\varepsilon(\alpha)$.

(2) $\Sigma^1 \cap (\alpha \cup b_1 \cup b_2) \subset (q_6,1)$ except for $\Sigma^1 \cap (b_1 \cup b_2)$.

(3) $\Sigma^1 \cap \text{Bd } K_1' \subset a$.

We next choose N so large that, if $i \geq N$, Σ_i satisfies several conditions--the first being: Σ_i intersects $b_1' \cup b_2'$ inside S_6 and $b_1 \cup b_2$ inside Σ^1. Secondly, Σ_i will be such that $\Sigma_i \subset N_\varepsilon(\alpha)$ and $\Sigma_i \subset \text{Bd } K_1' \subset a$. Let $\{B^i\}_{i=2}^6$ be disks on $\cup_{i=2}^6 S_i$ with the following properties:

(1) Each B^i is the union of two disks B_0^i and B_1^i where $B_0^i \subset D_0^i$ and $B_1^i \subset D_1^i$ and $B_0^i \cap \text{Bd } D_0^i = B_1^i \cap \text{Bd } D_0^i = B_1^i \cap \text{Bd } D_1^i$.

(2) $\mathrm{Bd}(B_1^i) - B_0^i \subset \mathrm{Ext\ Bd}\ K_1'$.

(3) Int B^i contains d_i.

(4) $(\cup B^i) \subset \mathrm{Int}\ \Sigma^1$.

Note that this requires Bd $B^i \cap \alpha = \emptyset$. Finally, N is chosen so that for $j \geq N$, $\Sigma_j \cap D_0^i \subset \mathrm{Int}\ B_0^i$ and Bd $B^i \cap$ (3-cell bounded by Σ_j) $= \emptyset$. $i = 2,\cdots,6$. We assume the following claim (see [5], claim II).

Claim. $\Sigma_n \cap \mathrm{Bd}\ K_1'$ may be assumed to be a finite collection of disjoint circles which bound disjoint disks E_j, $j = 1,\cdots,m$ on Σ_n with Int $E_j \subset$ Int Bd K_1' and which link α on Bd K_1', i.e., the small disks on α bounded by $\{\mathrm{Bd}\ E_j\}_{j=1}^m$ contain α. Further, these circles hit $b_1 \cup b_1'$ and Bd D_0^i, $i = 2,\cdots,6$, in two points each. Also $S_i \cap E_j$ is a simple arc $e_{i,j}$ for each i, $i = 2,\cdots,6$, and j, $j = 1,\cdots,m$.

Now we must adjust $\Sigma_n \cap (\cup_{i=2}^6 S_i)$ in several special cases with all necessary adjustments being made within $N_\varepsilon(\alpha)$. We will assume that $\Sigma_n \cap (\cup_2^6 S_i)$ is a finite collection of disjoint circles since each sphere is tame. Let $G \subset \Sigma_n \cap S_4$ be an innermost circle on Σ_n. Let $\Sigma_n(G)$ denote the innermost disk it bounds on Σ_n and $S_4(G)$ the disk it bounds on $S_4 - s_4$. The adjustments are made in exactly the same manner as before ([5]), and will show the existence of a disk $H \subset \Sigma_n$ which hits either $b_1 \cup b_2$, or $b_1' \cup b_2'$ in exactly two points. H is then used along with either Σ^1 or S_1, depending on whether H comes from Case I or Case II ([5]), to construct a sphere which contains α and hits $b_1 \cup b_2 \cup b_1' \cup b_2'$ in exactly four points. As stated before this sphere shows the existence of the desired annulus outside K. Since K is a 3-cell, a similar annulus would exist inside K. Furthermore, the annulus inside K could be chosen to have the same boundary components a_1 and a_2 as A_1, the one on the outside. The union of these two annuli would yield a torus. In like manner, a sequence of pairs of annuli and therefore a sequence of tori could be chosen. By construction, it is obvious that the sequence of tori could be chosen so that their intersection is s and so that they are concentric. This completes the proof.

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