TOPOLOGY PROCEEDINGS
Volume 5, 1980
Pages 77–104

http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

In 1961, J. Ceder [C] defined the M_i-spaces, $i = 1,2,3$, proved that $M_1 \Rightarrow M_2 \Rightarrow M_3$, and asked whether any of the implications reversed. H. Junnila [J] and the author [G_1] independently proved that M_3-spaces, usually called "stratifiable spaces," are M_2. The question remains whether $M_3 \Rightarrow M_1$, that is, whether every stratifiable space has a σ-closure-preserving base. Some partial results obtained so far are that the closed image of a metric space is M_1 (F. Slaughter [Sl]), and that σ-discrete stratifiable spaces are M_1 [G_2]. Recently, R. Heath and Junnila showed that every stratifiable space is the image of an M_1-space under a perfect retraction (and hence is a closed subset of an M_1-space).

Let us call a space which is a countable union of closed metrizable subspaces an F_σ-metrizable space. In the first part of this paper, we prove that every stratifiable F_σ-metrizable space is M_1. Many common examples of stratifiable spaces seem to be of this type. For example, all the examples given in Ceder's paper, as well as the chunk-complexes (which he proves to be M_1), are F_σ-metrizable. Hyman's M-spaces [H], called paracomplexes and proved M_1 by J. Nagata [N], are also of this type.

Another interesting class of stratifiable spaces is the following. Let I be an index set, and for each $i \in I$, let X_i be a stratifiable space. Let $p \in \square_{i \in I} X_i$, where "$\square$" denotes the box product. Let $Y = \{x \in \square_{i \in I} X_i : x(i) = p(i)$ for all but finitely many $i \in I\}$. Borges $[B_3]$ proved that if each X_i is stratifiable, so is Y. It is not hard to show that if each X_i is F_σ-metrizable, then so is Y; hence Y is M_1 in this case.

In $[G_2]$, we asked whether a stratifiable space which has a σ-discrete network consisting of compact sets is M_1. Since compact stratifiable spaces are metrizable, our result implies an affirmative answer to this question.

Unfortunately, the class of stratifiable F_σ-metrizable spaces is not closed under closed maps. In fact, the closed image of a metric space need not be F_σ-metrizable [F]. It must be M_1, though, by Slaughter's result mentioned above. Now suppose a space X has the following property: whenever H and K are closed subsets of X with $H \subset K$, then H has a σ-closure-preserving outer base in K (i.e., there is a σ-closure-preserving collection $\mathcal{U}$ of relatively open subsets of K such that whenever $H \subset O$, O open, there exists $U \in \mathcal{U}$ with $H \subset U \subset O$). In the second section of this paper, we prove that if an M_1-space X has the above property, then every closed image of X is M_1 and has the same property. From the fact that stratifiable F_σ-metrizable spaces are M_1, it is easily shown that they also have the above property. Thus every closed image of a stratifiable F_σ-metrizable space is M_1. This generalizes Slaughter's theorem, and answers a question of Nagata concerning the paracomplexes mentioned above.

Nagata also showed that if X is a paracomplex, then Ind $X \leq n$ if and only if X has a σ-closure-preserving base $\mathcal{B}$ such that $\mathrm{Ind}(\partial B) \leq n - 1$ for every $B \in \mathcal{B}$. He asked if this result is true for any M_1-space. Mizokami [M] showed that it is true for an M_1-space which is F_σ-metrizable and satisfies a certain further condition. With our techniques, we can show it is true for any stratifiable F_σ-metrizable space.

2. Definitions and Other Preliminaries

All spaces are assumed to be regular. Let A^o denote the interior of a set A. A collection $\mathcal{G}$ of subsets of a space X is *interior-preserving* if whenever $\mathcal{G}' \subset \mathcal{G}$, then $(\cap\mathcal{G}')^o = \cap\{G^o: G \in \mathcal{G}'\}$. A collection $\mathcal{H}$ of subsets of X is *closure-preserving* if whenever $\mathcal{H}' \subset \mathcal{H}$, then $\overline{\cup\mathcal{H}'} = \cup\{\overline{H}: H \in \mathcal{H}'\}$. It is easy to see that the set of complements of an interior-preserving family is closure-preserving, and vice-versa.

If H is a subset of a space X, an *outer base* for H is a collection $\mathcal{U}$ of open subsets of X such that whenever H is contained in an open set O, then there exists $U \in \mathcal{U}$ such that $H \subset U \subset O$.

A collection $\mathcal{B}$ is a *quasi-base* for X if whenever $x \in U$, U open, there exists $B \in \mathcal{B}$ such that $x \in B^o \subset B \subset U$. ($B^o$ denotes the interior of B.) A space X is an M_1-*space* (M_2-*space*) if X has a σ-closure-preserving base (quasi-base). An M_3-*space*, or *stratifiable space*, is the same as an M_2-space.

We will often use the following characterization of M_2-spaces due to Nagata [N_2].

Theorem 2.1 (Nagata). *A regular space* X *is an* M_2-*space if and only if for each* $x \in X$ *and* $n \in \omega$, *there exists an open neighborhood* $g_n(x)$ *of* x *such that*

(1) $y \in g_n(x) \Rightarrow g_n(y) \subset g_n(x)$; *and*

(2) if H *is closed and* $x \notin H$, *there exists* $n \in \omega$ *such that* $x \notin \overline{\bigcup_{x \in H} g_n(x)}$.

Clearly, we may assume $g_0(x) \supset g_1(x) \cdots$.

A collection $\mathcal{F}$ is a *network* for X if whenever $x \in U$, U open, there exists $F \in \mathcal{F}$ such that $x \in F \subset U$. X is a σ-*space* if X has a σ-discrete network.

A space X is *monotonically normal* if for every pair (H,K) of disjoint closed subsets, there exists an open set D(H,K) such that

(1) $H \subset D(H,K) \subset \overline{D(H,K)} \subset X - K$;

(2) $H \subset H'$ and $K \supset K' \Rightarrow D(H,K) \subset D(H',K')$.

We shall be using the fact that stratifiable spaces are paracompact and perfectly normal [C], that they are σ-spaces [H], and that they are monotonically normal [HLZ]. Also, every subspace of a stratifiable space is stratifiable [C], and every closed image of a stratifiable space is stratifiable [B].

3. Main Results, Outlines of the Proofs, and Some Questions

Since the proof of our main results are rather long and tedious, we will defer them to later sections, giving only brief outlines here.

Theorem 3.1. *Let* X *be stratifiable and* F_σ*-metrizable. Then* X *is* M_1. *Also,* Ind X $\leq$ n *if and only if* X *has a* σ*-closure-preserving base* $\mathcal{B}$ *such that* Ind$(\partial B) \leq n-1$ *for each* $B \in \mathcal{B}$.

Outline of proof. Suppose X is stratifiable. Then there exist $g_n(x)$'s satisfying the conditions of Theorem 2.1. The standard way to get a σ-closure-preserving quasi-base back from the $g_n(x)$'s is as follows. For each closed set H, define $G_n(H) = \bigcup_{x \in H} g_n(x)$. It is easy to see from property (1) of Theorem 2.1 that

$$\mathcal{G}_n = \{G_n(H)\colon H \text{ is closed, } H \subset X\}$$

is interior-preserving. Hence,

$$\mathcal{B}_n = \{X - G_n(H)\colon H \text{ closed, } H \subset X\}$$

is closure-preserving, and from property (2), $\bigcup_{n \in \omega} \mathcal{B}_n$ is a closed quasi-base. A naive attempt to get a σ-closure-preserving base would be to define

$$\mathcal{B}_n' = \{(X - G_n(H))^{\circ}\colon H \text{ closed, } H \subset X\}.$$

However, $\mathcal{B}_n'$ may fail to be closure-preserving. But it turns out that if the $G_n(H)$'s are regular open sets, then $\mathcal{B}_n'$ will be closure-preserving. (See Lemma 5.1.)

So we construct $g_n(x)$'s satisfying the conditions of Theorem 2.1 so that the corresponding $G_n(H)$'s are regular open. We do this by constructing a certain sequence $\mathcal{V}_0, \mathcal{V}_1, \cdots$ of locally finite open covers of X, and then use the $\mathcal{V}_i$'s to construct the $g_n(x)$'s, so that:

(i) each $g_n(x)$ is an element of some $\mathcal{V}_i$;

(ii) for each $m \in \omega$, every union of elements of $\bigcup_{i \leq m} \mathcal{V}_i$ is regular open; and

(iii) if H is closed and $y \notin \bigcup_{x \in H} g_n(x)$, then there exists an integer k such that

$$y \notin Cl(\bigcup\{g_n(x) : x \in H,\ g_n(x) \notin \bigcup_{i \leq k} \mathcal{V}_i\}).$$

It easily follows that $G_n(H) = \bigcup_{x \in H} g_n(x)$ is regular open whenever H is closed. The F_σ-metrizable hypothesis is used to obtain property (iii). It is possible to construct $\mathcal{V}_0, \mathcal{V}_1, \cdots$ and the $g_n(x)$'s satisfying (i) and (ii) in any stratifiable space.

The "if" part of the last statement of Theorem 3.1 is a result of Nagata [N_2]. To obtain the "only if" part, we show that if Ind X $\leq$ n, we can construct the $\mathcal{V}_i$'s so that Ind(∂V) $\leq$ n-1 for each V $\in \mathcal{V}_i$. It then follows that Ind(∂B) $\leq$ n-1 for each B $\in \mathcal{B}_n'$.

Theorem 3.2. *Suppose* X *is stratifiable and has the following property: whenever* H *and* K *are closed subsets of* X *with* H $\subset$ K, *then* H *has a σ-closure-preserving outer base in* K. *Then every closed image of* X *has the same property, and is therefore* M_1.

Outline of proof. Let (*) denote the property of Theorem 3.2. Any stratifiable space satisfying (*) is M_1. This follows easily from the facts that every closed subset has a σ-closure-preserving outer base, and that stratifiable spaces are σ-spaces.

Let f be a closed map of X onto Y, where X is stratifiable and satisfies (*). Since every closed subset K of Y is the closed image of a stratifiable space satisfying (*), namely $f^{-1}(K)$, it is enough to show that every closed image of a stratifiable space satisfying (*) has the property that every closed subset has a σ-closure-preserving outer base. So we are done if we show that Y has this property.

By a theorem of Okuyama [O], $Y = Y' \cup Y''$, where $f^{-1}(y)$ is compact for each $y \in Y'$, and Y'' is σ-discrete. We use this to show that $Y = Y_o \cup Y_1$, where Y_o is a closed irreducible image of a closed subset X_o of X, and Y_1 is open and σ-discrete. From results in [BL], it follows that every closed subset of Y_o has a σ-closure-preserving outer base in Y_o. Thus Y can be written as the union of a closed subspace having the property we want, and an open σ-discrete subspace. The final step is to show that any stratifiable space which admits such a decomposition also has the property that every closed subset has a σ-closure-preserving outer base.

Remark. For stratifiable spaces, property (*) is equivalent to the following property: whenever H and K are closed subsets of X with $H \subset K$, then K/H is M_1.

Corollary 3.3. *The closed image of a stratifiable F_σ-metrizable space is* M_1.

Proof. Suppose X is stratifiable and F_σ-metrizable. Let $H \subset K \subset X$, where H and K are closed. Then K/H is stratifiable and F_σ-metrizable, hence M_1. By the above remark, X satisfies the conditions of Theorem 3.2.

It is not known whether every M_1-space satisfies the property of Theorem 3.2. In fact, it is not known if every closed subset of an M_1-space is M_1. However, the result of Heath and Junnila mentioned in the introduction implies, as they note, that this question is equivalent to the $M_3 \Rightarrow M_1$ question. It is also not known whether every closed subset of an M_1-space has a σ-closure-preserving outer base. But if not, then by results of Ceder, there is a stratifiable space which is not M_1. On the other hand, Borges and Lutzer [BL] have shown that if each point of a stratifiable space has a σ-closure-preserving base, then every stratifiable space is M_1.

Borges and Lutzer have also shown that if every closed subspace of a space X is M_1, then every perfect image of X is M_1. This suggests the following question, which would generalize Theorem 3.2 if answered affirmatively.

Question 3.4. If every closed subspace of a space X is M_1, is every closed image of X also M_1?

A class of spaces which Heath and Junnila called M_o-spaces may have an important role to play in settling the $M_3 \Rightarrow M_1$ question. An M_o-*space* is a space which has a σ-closure-preserving base of open and closed sets. It is easy to see that every subspace of an M_o-space is M_o. Thus every perfect image of an M_o-space is M_1. Recently, Junnila [J_2] has obtained an alternate proof that stratifiable F_σ-metrizable spaces are M_1 by showing that they are perfect images of M_o-spaces. Heath and Junnila ask whether every stratifiable space is the perfect image of an M_o-space. If so, then $M_3 \Rightarrow M_1$. Our next corollary shows that it would be enough (for the purpose of obtaining $M_3 \Rightarrow M_1$) to prove that every stratifiable space is the closed image of an M_0-space.

Corollary 3.5. *The closed image of an* M_0*-space is* M_1.

Proof. We show that every M_0-space X satisfies the property of Theorem 3.2. If $K \subset X$, then K is M_0. By mimicing Ceder's proof that every closed subset of an M_2-space has a closure-preserving outer quasi-base, we see that every closed subset of K has a closure-preserving base in K.

The class of stratifiable F_σ-metrizable spaces is not closed under closed maps or countable products. These are still M_1, of course. In fact, since products of perfect maps are perfect, the countable product X of stratifiable F_σ-metrizable spaces is the perfect image of an M_0-space. Hence X satisfies the property of Theorem 3.2, and so every closed subspace and closed image of X is M_1. But further iterations of the procedures of taking closed subspaces, closed images, and countable products, produces spaces that I can't prove are M_1. What one might aim for is a solution to the following:

Problem 3.6. Find a class of M_1-spaces which contains the F_σ-metrizable spaces, and which is closed under closed subspaces, closed images, and countable products.

Note that the class of stratifiable F_σ-metrizable spaces is closed under arbitrary subspaces, perfect images, and finite products. The class of stratifiable spaces satisfying the property of Theorem 3.2 is closed under closed subspaces and closed images. Thus one would have a solution to the Problem 3.6 if one could show that this property is closed under countable products. For another approach, note that the class of perfect images of M_o-spaces satisfies all the desired properties except perhaps closure under closed images.

Although we don't do it here, the techniques of section 6 can be used to show that if a stratifiable space X is the union of a closed M_1-space and an F_σ-metrizable space, then X is M_1. This suggests a couple of questions, for which affirmative answers to both (or a negative answer to one) would obviously settle the $M_3 \Rightarrow M_1$ question.

Question 3.7. If a stratifiable space X is a countable union of a closed M_1 subspaces, is X M_1?

Question 3.8. Is every stratifiable space the countable union of closed M_1 subspaces?

4. Preliminary Lemmas

In this section we present a series of lemmas on regular open sets, leading up to the result that in a paracompact hereditarily normal space, every open cover has a locally finite refinement $\mathcal{V}$ such that every union of elements of $\mathcal{V}$ is regular open. In fact, if $\mathcal{W}$ is any locally finite collection such that every union of elements of $\mathcal{W}$ is regular open, then $\mathcal{V}$ can be constructed so that every union of elements of $\mathcal{V} \cup \mathcal{W}$ is regular open.

Lemma 4.1. *Let* X *be a hereditarily normal space. Suppose* U, V, *and* $U \cup V$ *are regular open, and* H *is a relatively closed subset of* V. *Suppose* $\overline{H}$ *is contained in an open set* O. *Then there is a set* W *such that* $H \subset W \subset V$, $\overline{W} \subset O$, *and both* W *and* $U \cup W$ *are regular open. If* X *is perfectly normal,* $\operatorname{Ind} X \leq n$, *and* $\operatorname{Ind}(\partial U) \leq n-1$, *then we can obtain* $\operatorname{Ind}(\partial W) \leq n-1$.

Proof. Let O' be an open set such that $\overline{H} \subset O' \subset \overline{O}' \subset O$. Using the hereditary normality X, we can find open sets V_1 and V_2 such that

$$H \subset V_1 \subset \overline{V}_1^V \subset V_2 \subset \overline{V}_2^V \subset V \cap O' ,$$

where $\overline{A}^V$ denotes the closure of A in the subspace V. Now let $W = \overline{V_1 \cup U}^{\circ} \cap \overline{V}_2^{\circ}$. We claim that W has the desired properties.

Clearly, $H \subset W \subset V$, and $\overline{W} \subset O$. Also, since the intersection of two regular open sets is regular open, W is regular open. It remains to show $W \cup U$ is regular open. To see this, suppose $p \in \overline{W \cup U}^{\circ} - W \cup U$. Observe that $\overline{W \cup U} \subset \overline{V_1 \cup U}$. Thus $p \in \overline{V_1 \cup U}^{\circ} - U$. Hence $p \in \overline{V}_1$, since U is regular open. Also, $p \in \overline{V \cup U}^{\circ} - U = (V \cup U) - U$, so $p \in V$. Hence $p \in \overline{V}_1 \cap V \subset V_2$. So we have $p \in \overline{V_1 \cup U}^{\circ} \cap \overline{V}_2^{\circ} = W$, contradiction.

To see the last statement, note that $\partial W \subset \partial V_1 \cup \partial V_2 \cup \partial U$. Since $\operatorname{Ind} V \leq n$, we can obtain $\operatorname{Ind}(\partial_V V_i) \leq n-1$, $i = 1,2$, where "∂_V" denotes the boundary of a set in the subspace V. Now $\partial V_i = [\partial V_i \cap \partial V] \cup \partial_V V_i$. Thus $\operatorname{Ind}(\partial V_i) \leq n-1$, and so $\operatorname{Ind}(\partial V_1 \cup \partial V_2 \cup \partial U) \leq n-1$. Thus $\operatorname{Ind}(\partial W) \leq n-1$.

Lemma 4.2. *If* $\mathcal{U}$ *is a collection of subsets of a space* X *such that every finite union of elements of* $\mathcal{U}$ *is regular open, then every finite union of finite intersections of elements of* $\mathcal{U}$ *is regular open.*

Proof. This follows from the fact that a finite union of finite intersections of elements of $\mathcal{U}$ can be written as the finite intersection of finite unions of elements of $\mathcal{U}$, and that a finite intersection of regular open sets is regular open.

Lemma 4.3. *Suppose* $\mathcal{U}$ *is a finite collection of open sets of a hereditarily normal space* X *such that every union of elements of* $\mathcal{U}$ *is regular open. Let* $H \subset O$, H *closed,* O *open. Then there exists a set* W *such that* $H \subset W \subset O$, *and every union of elements of* $\mathcal{U} \cup \{W\}$ *is regular open. If* X *is perfectly normal,* $\mathrm{Ind}\, X \leq n$, *and* $\mathrm{Ind}(\partial U) \leq n-1$ *for each* $U \in \mathcal{U}$, *then we can obtain* $\mathrm{Ind}(\partial W) \leq n-1$.

Proof. If $|\mathcal{U}| = 1$, apply Lemma 1 with U the element of $\mathcal{U}$, $V = X$, and H and O as in the hypothesis of this lemma. The set W guaranteed by Lemma 4.1 is easily seen to satisfy the desired conditions.

Now suppose $|\mathcal{U}| = n > 1$. Let $\mathcal{U} = \{U_0, U_1, \cdots, U_{n-1}\}$. Let W_0 be a regular open set such that $H \subset W_0 \subset \overline{W}_0 \subset O$, and $(\cup\mathcal{U}) \cup W_0$ is regular open.

Suppose W_k has been defined, $k < n$. For each subset I of $k + 1$ distinct elements of $n = \{0,1,\cdots,n-1\}$, let $H(I) = (\partial W_k) \cap (\bigcap_{i\in I} U_i)$. Then the hypotheses of Lemma 4.1 are satisfied with $U = \bigcup_{j\notin I} U_j$, $V = \bigcap_{i\in I} U_i$, $H = H(I)$, and O as in this lemma. (Note the use of Lemma 4.2 to get $U \cup V$

regular open.) Let $W(I)$ be the set given by Lemma 4.1, and let

$$W_{k+1} = W_k \cup (\cup\{W(I) : I \subset n, |I| = k + 1\}).$$

Let $W = W_n$. We claim that W has the desired property. To see this, assume $\mathcal{M} \subset \mathcal{U} \cup \{W\}$, and $x \in \overline{\cup\mathcal{M}}^{o} - \cup\mathcal{M}$. Clearly, we may assume $W \in \mathcal{M}$ and $x \in \overline{W}$.

Let $I = \{i < n : x \in U_i\}$, and let $m = |I|$. If $m = 0$, then $x \in \overline{(\cup\mathcal{U}) \cup W}^{o} - (\cup\mathcal{U}) \cup W = \overline{(\cup\mathcal{U}) \cup W_o}^{o} - (\cup\mathcal{U}) \cup W_o$, contradiction. So we may assume $m \geq 1$. Then $x \notin \overline{W}_{m-1}$, for otherwise $x \in \partial W_{m-1} \cap (\bigcap_{i \in I} U_i) = H(I) \subset W(I) \subset W$. So there is an open set G such that

$$x \in G \subset \overline{(\cup\mathcal{M})}^{o} \cap (\bigcap_{i \in I} U_i) \cap (X - \overline{W}_{m-1}).$$

Since $W(I) \cup (\bigcup_{j \notin I} U_j)$ is regular open and doesn't contain x, there exists $y \in G$ with $y \notin \overline{W(I) \cup (\bigcup_{j \notin I} U_j)}$. But $y \in \overline{\cup\mathcal{M}}$, so $y \in \overline{\cup(\mathcal{M} - \{W\})}$ or $y \in \overline{W}$. Since $\cup(\mathcal{M} - \{W\}) \subset \bigcup_{j \notin I} U_j$, it must be true that $y \in \overline{W}$. Thus there exists a nonempty set $J \subset n$ such that $y \in \overline{W(J)}$. Since $y \notin \overline{W}_{m-1}$, we have $|J| \geq m$. Since $y \notin \overline{W(I)}$, we have $J \neq I$. Therefore, there exists $j_o \in J - I$. But $\overline{W(J)} \subset \overline{\bigcap_{i \in J} U_i} \subset \overline{U}_{j_o}$, contradicting $y \notin \overline{\bigcup_{j \notin I} U_j}$.

Thus W has the desired property. The last statement of Lemma 4.3 follows because W is the union of a finite number of sets obtained from the application of Lemma 4.1.

Lemma **4.4.** *Let $\mathcal{U}$ be an open cover of a paracompact hereditarily normal space X, and let $\mathcal{V}$ be a locally finite collection such that every union of elements of $\mathcal{V}$*

is regular open. Then there is a locally finite refinement $\mathcal{W}$ *of* $\mathcal{U}$ *such that every union of elements of* $\mathcal{V} \cup \mathcal{W}$ *is regular open. If* X *is perfectly normal,* $\operatorname{Ind} X \leq n$, *and* $\operatorname{Ind}(\partial V) \leq n-1$ *for each* $V \in \mathcal{V}$, *then we can obtain* $\operatorname{Ind}(\partial W) \leq n-1$ *for each* $W \in \mathcal{W}$.

Proof. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a locally finite open cover of X by sets whose closures refine $\mathcal{U}$ and meet only finitely many elements of $\mathcal{V}$, and such that each $\mathcal{F}_n$ is discrete. Let $\mathcal{G} = \{G_F: F \in \mathcal{F}\}$ be a shrinking of $\mathcal{F}$; i.e., $\mathcal{G}$ is an open cover such that $\overline{G}_F \subset F$ for each $F \in \mathcal{F}$. Let $\mathcal{V}(F) = \{V \in \mathcal{V}: V \cap F \neq \emptyset\}$. For each $F \in \mathcal{F}_o$, let W_F be the set given by Lemma 4.3 where $\mathcal{U} = \mathcal{V}(F)$, $H = \overline{G}_F$, and $O = F$. Then $\overline{G}_F \subset W_F \subset F$, and every union of elements of $\mathcal{V} \cup \{W_F: F \in \mathcal{F}_o\}$ is regular open. Let $\mathcal{W}_o = \{W_F: F \in \mathcal{F}_o\}$.

Now suppose that for each $F \in \mathcal{F}_k$, $k < n$, we have defined a W_F such that

(i) $\mathcal{W}_k = \{W_F: F \in \mathcal{F}_k\}$ is a discrete collection of open sets;

(ii) if $F \in \mathcal{F}_k$, then $\overline{G}_F - \bigcup_{j<k} (\cup\mathcal{W}_j) \subset W_F \subset F$;

(iii) if $F \in \mathcal{F}_k$, then $W_F \cap G_{F'} = \emptyset$ whenever $F' \in \mathcal{F}_j$ with $j < k$;

(iv) every union of elements of $\mathcal{V} \cup (\bigcup_{j\leq k} \mathcal{W}_j)$ is regular open.

To define W_F for $F \in \mathcal{F}_n$, let $\mathcal{O} = \{O_F: F \in \mathcal{F}_n\}$ be a discrete collections of open sets such that

$$\overline{G}_F - \bigcup_{k<n} (\cup\mathcal{W}_k) \subset O_F \subset F, \text{ and}$$

$$O_F \cap (\cup\{G_F: F \in \mathcal{F}_k, k < n\}) = \emptyset.$$

This is possible, since $\cup\{\overline{G}_F: F \in \mathcal{F}_k, k < n\} \subset \bigcup_{k<n} (\cup\mathcal{W}_k)$. Now, for each $F \in \mathcal{F}_n$, let W_F be the set given by Lemma 4.3 applied to the case where $\mathcal{U} = \mathcal{V}(F) \cup \{\cup\mathcal{W}_k: k < n\}$, $H = \overline{G}_F - \bigcup_{k<n} (\cup\mathcal{W}_k)$, and $O = O_F$.

It is easy to check that $\mathcal{W}_n = \{W_F: F \in \mathcal{F}_n\}$ satisfies properties (i)-(iii) above. To see that (iv) is satisfied, suppose $x \in \overset{\circ}{\overline{\cup\mathcal{M}}} - \cup\mathcal{M}$, where $\mathcal{V} \cup (\bigcup_{k\leq n} \mathcal{W}_k)$. Since this collection is locally finite, we may assume $\mathcal{M}$ is finite and $x \in \overline{M} - M$ for every $M \in \mathcal{M}$. Let $I = \{k < n: \mathcal{M} \cap \mathcal{W}_k \neq \emptyset\}$. For each $i \in I$, $x \notin \cup\mathcal{W}_i$ since $\mathcal{W}_i$ is discrete and $x \in \overline{W} - W$ for some $W \in \mathcal{W}_i$. By the induction hypothesis, we can assume there exists $W_F \in \mathcal{M} \cap \mathcal{W}_n$, where $F \in \mathcal{F}_n$. Then $\mathcal{M} \cap \mathcal{V} = \mathcal{M} \cap \mathcal{V}(F)$, since $x \in \overline{W}_F \subset F$. Thus x is in the interior of the closure of

$$\cup[(\mathcal{M} \cap \mathcal{V}(F)) \cup \{\cup\mathcal{W}_i: i \in I\} \cup \{W_F\}],$$

but is not in this set, contradicting the way W_F was defined (i.e., the above set must be regular open). Thus $\mathcal{W}_n$ satisfies (iv).

Let $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$. That $\mathcal{W}$ is a refinement of $\mathcal{U}$ follows from property (ii). That $\mathcal{W}$ is locally finite follows from (i) and (iii). And that every union of elements of $\mathcal{V} \cup \mathcal{W}$ is regular open follows from (iv) and the local finiteness.

Let us see how to obtain the last statement of Lemma 4.4. First note that under the hypotheses we can get $\operatorname{Ind}(\partial W) \leq n-1$ for $W \in \mathcal{W}_o$, since these sets were also obtained from Lemma 4.3. If it's true for $W \in \bigcup_{k<n} \mathcal{W}_k$, then it's also true for $W \in \mathcal{W}_n$, since these sets were also

obtained from Lemma 4.3, with the $\mathcal{U}$ of Lemma 4.3 being a subset of $\mathcal{V}$ together with $\{\cup\mathcal{W}_k: k < n\}$, and $\mathrm{Ind}(\partial(\cup\mathcal{W}_k)) \leq n-1$ for $k < n$ since each $\mathcal{W}_k$ is discrete.

5. Stratifiable F_σ-Metrizable Spaces

In this section, we prove that stratifiable F_σ-metrizable spaces are M_1. First, an easy lemma.

Lemma 5.1. *Let $\mathcal{U}$ be an interior-preserving collection of regular open subsets of a space* X. *Then* $\{(X - U)^o: U \in \mathcal{U}\}$ *is closure-preserving.*

Proof. Suppose $\mathcal{U}' \subset \mathcal{U}$, and $x \in \overline{\cup\{(X - U)^o: U \in \mathcal{U}'\}}$. Suppose for each $U \in \mathcal{U}'$, we have $x \notin \overline{(X - U)^o}$. Since each U is regular open, we must have $x \in U$ for each $U \in \mathcal{U}'$. Thus $x \in \cap\mathcal{U}'$, which is an open set missing $(X - U)^o$ for every $U \in \mathcal{U}'$. This contradiction establishes the lemma.

Proof of Theorem 3.1. Let X be a stratifiable F_σ-metrizable space. Let $X = \bigcup_{n\in\omega} M_n$, where each M_n is a closed, metrizable subspace of X. Let $M_n = \bigcap_{m\in\omega} O_{n,m}$, where $O_{n,m}$ is open.

Let $\{\mathcal{U}'_{n,m}\}_{m\in\omega}$ be a sequence of relatively open covers of M_n which is a development for M_n. For each $U' \in \mathcal{U}'_{n,m}$, let U be open in X such that $U \cap M_n = U'$, and $U \subset O_{n,m}$. Let $\mathcal{U}_{n,m} = \{U: U' \in \mathcal{U}'_{n,m}\}$.

For each $x \in X$ and $n \in \omega$, let $g_n(x)$ be an open neighborhood of x such that the $g_n(x)$'s satisfy the conditions of Theorem 2.1, and that $g_o(x) = X$ for each $x \in X$. Let $\mathcal{F} = \bigcup_{n\in\omega} \mathcal{F}_n$ be a closed network for X such that each $\mathcal{F}_n$ is discrete. For each $F \in \mathcal{J}_n$, let U_F be an open set such that $F \subset U_F$, and $U_F \cap U_{F'} = \emptyset$ whenever $F' \in \mathcal{J}_n$, $F' \neq F$. Let $V_F = U_F \cap (\cap\{g_i(x): i \leq n \text{ and } F \subset g_i(x)\})$. Since the $g_i(x)$'s satisfy property (1) of Theorem 2.1, V_F is open.

Now use Lemma 4.4 to inductively construct a sequence $\mathcal{V}_0, \mathcal{V}_1, \cdots$ of locally finite open covers of X such that

(i) $\mathcal{V}_{n+1}$ star-refines $\mathcal{V}_n$;

(ii) $\mathcal{V}_n$ refines $\{V_F: F \in \mathcal{J}_i\} \cup \{X - \cup\mathcal{J}_i\}$ for each $i \leq n$;

(iii) $\mathcal{V}_n$ refines $\mathcal{U}_{i,j} \cup \{X - M_i\}$ for each $i,j \leq n$;

(iv) every union of elements of $\bigcup_{i \leq n} \mathcal{V}_i$ is regular open.

It is easy to see from Lemma 4.2 that we may assume that if $x \in X$, then $\cap\{V \in \mathcal{V}_i: x \in V, i \leq n\}$ is an element of $\mathcal{V}_n$.

Claim I. *For each* $x \in X$, *for each* $n \in \omega$, *there exists* $m \in \omega$ *such that* $st(x,\mathcal{V}_m) \subset g_n(x)$.

Proof of Claim I. There exists $n' \geq n$ and $F \in \mathcal{J}_{n'}$ such that $x \in F \subset g_n(x)$. (To get $n' \geq n$, we can assume that each $\mathcal{J}_n$ is repeated infinitely often.) If $x \in V \in \mathcal{V}_{n'}$, then by (ii), $V \subset V_F = U_F \cap (\cap\{g_i(y): i \leq n' \text{ and } F \subset g_i(y)\}) \subset g_n(x)$.

Let $m = n' + 1$. Then $st(x,\mathcal{V}_m)$ is contained in some element of $\mathcal{V}_{n'}$, so $st(x,\mathcal{V}_m) \subset g_n(x)$.

Claim II. *If* H *is closed in* X, *and* $y \notin H \cap M_n$, *then there exists* $m \in \omega$ *and* $V \in \mathcal{V}_m$ *such that* $y \in V$ *and* $st(V,\mathcal{V}_m) \cap H \cap M_n = \emptyset$.

Proof of Claim II. If $y \notin M_n$, there exists n' such that $y \notin O_{n,n'}$. Let $m = n + n' + 1$, and pick $V \in \mathcal{V}_m$ with $y \in V$. Then $st(V,\mathcal{V}_m) \subset W \in \mathcal{V}_{n+n'}$. W must be contained in some element of $\mathcal{U}_{n,n'} \cup \{X - M_n\}$. Each element of $\mathcal{U}_{n,n'}$ is contained in $O_{n,n''}$, so $W \subset X - M_n$. Thus $st(V,\mathcal{V}_m) \cap H \cap M_n = \emptyset$.

If $y \in M_n$, there exists $n'' \in \omega$ such that $st(y,\mathcal{U}_{n,n''}) \cap H \cap M_n = \emptyset$. Let $m = n + n'' + 1$, and pick $V \in \mathcal{V}_m$ with $y \in V$. Then $st(V,\mathcal{V}_m) \subset W \in \mathcal{V}_{n+n''}$. But $W \subset U$ for some $U \in \mathcal{U}_{n,n''}$, where $y \in U \cap M_n \in \mathcal{U}'_{n,n''}$. Thus $U \cap H \cap M_n = \emptyset$, so $st(V,\mathcal{V}_m) \cap H \cap M_n = \emptyset$.

For each $x \in X$, let $j(x) \in \omega$ be such that $x \in M_{j(x)} - \bigcup_{i<j(x)} M_i$. From Claims I and II, it is easy to see that, for each $n \in \omega$, there is a least integer $l(n,x)$ such that $st(x,\mathcal{V}_{l(n,x)}) \subset g_n(x) - \bigcup_{i<j(x)} M_i$. We define $g'_n(x) = \bigcap\{V \in \mathcal{V}_i : x \in V \in \mathcal{V}_i,\ i \le l(j(x) + n, x)\}$. By the statement immediately preceding Claim I we have $g'_n(x) \in \mathcal{V}_{l(j(x)+n,x)}$. It's not really necessary to have this, but we said we could get it in the outline in section 3.

Claim III. *If* $y \in g'_n(x)$, *then* $g'_n(y) \subset g'_n(x)$.

Proof of Claim III. If $y \in g'_n(x)$, then $y \in g_{j(x)+n}(x) \subset g_n(x)$, so $g_n(y) \subset g_n(x)$. Since $g'_n(x) \cap (\bigcup_{i<j(x)} M_i) = \emptyset$, we have $j(y) \ge j(x)$. Thus $g_{j(y)+n}(y) - \bigcup_{i<j(y)} M_i \subset g_{j(x)+n}(x) - \bigcup_{i<j(x)} M_i$. If $l(j(y)+n,y) < l(j(x)+n,x)$, then we would have $st(x,\mathcal{V}_{l(j(y)+n,y)}) \subset st(y,\mathcal{V}_{l(j(y)+n,y)}) \subset g_{j(y)+n}(y) - \bigcup_{i<j(y)} M_i \subset g_{j(x)+n}(x) - \bigcup_{i<j(x)} M_i$. This contradicts the definition of $l(j(x)+n,x)$. Thus $l(j(y)+n,y) \ge l(j(x)+n,x)$, and so $g'_n(y) \subset g'_n(x)$.

For each closed set H, define $G_n(H) = \bigcup_{x \in H} g_n'(x)$. By Claim III, the collection $\{G_n(H)\colon H \text{ closed}, H \subset X\}$ is interior-preserving. Since $g_n'(x) \subset g_n(x)$, the $g_n'(x)$'s satisfy property (2) of Theorem 2.1, so if we define

$$\mathcal{B}_n = \{(X - G_n(H))^{\circ}\colon H \text{ closed}, H \subset X\}$$

then $\bigcup_{n \in \omega} \mathcal{B}_n$ is a base for X. By Lemma 5.1, each $\mathcal{B}_n$ is closure-preserving if $G_n(H)$ is always regular open. So we are finished after proving the next claim.

Claim IV. $G_n(H)$ *is regular open.*

Proof of Claim IV. Suppose $y \in \overline{G_n(H)}^{\circ} - G_n(H)$. There exists $k \in \omega$ such that $y \notin \bigcup_{x \in H} g_k(x)$. So, since $g_n'(x) \subset g_{j(x)+n}(x)$, $y \notin \mathrm{Cl}(\bigcup\{g_n'(x)\colon x \in H, j(x) \geq k\})$.

By Claim II, for each $j < k$, there exists $m_j \in \omega$ and $V_j \in \mathcal{V}_{m_j}$ such that $y \in V_j$ and $\mathrm{st}(V_j, \mathcal{V}_{m_j}) \cap H \cap M_j = \emptyset$. Let $V = \bigcap_{j<k} V_j$. Suppose $x \in H$, $j(x) < k$, and $g_n'(x) \cap V \neq \emptyset$. Now $g_n'(x)$ is contained in some $W \in \mathcal{V}_{l(j(x)+n,x)}$, and $W \cap V_{j(x)} \neq \emptyset$. If $l(j(x)+n,x) \geq m_{j(x)}$, then $W \subset \mathrm{st}(V_{j(x)}, \mathcal{V}_{m_{j(x)}})$. But $x \in W \cap H \cap M_{j(x)}$, so $x \in \mathrm{st}(V_{j(x)}, \mathcal{V}_{m_{j(x)}}) \cap H \cap M_{j(x)}$, contradiction. Thus $l(j(x)+n,x) < m_{j(x)}$.

Let $m = \sup_{j<k}\{m_j\}$. We see, then, that

$$y \notin \mathrm{Cl}(\bigcup\{g_n'(x)\colon x \in H, j(x) < k, l(j(x)+n,x) \geq m\}).$$

Combining this with the first paragraph, we have

$$y \notin \mathrm{Cl}(\bigcup\{g_n'(x)\colon x \in H, l(j(x)+n,x) \geq m\}).$$

Thus y is in the interior of the closure of those $g_n'(x)$'s which are elements of $\bigcup_{i<m} \mathcal{V}_i$. This contradicts the fact that unions of elements of $\bigcup_{i<m} \mathcal{V}_i$ are regular open, and the proof that X is M_1 is finished.

Now suppose Ind X $\leq$ n. We will show how to obtain Ind(∂B) $\leq$ n-1 for each B $\in \bigcup_{n\in\omega} \mathcal{B}_n$. Since $\partial(X - G_k(H))^{o} \subset \partial G_k(H)$, we will be done if we can get Ind($\partial G_k(H)$) $\leq$ n-1 for an arbitrary k $\in \omega$ and closed set H.

By Lemma 4.4, we can add the following to the list of properties of the sequence $\mathcal{V}_0, \mathcal{V}_1, \cdots$:

(v) For each n $\in \omega$ and V $\in \mathcal{V}_n$, Ind(∂V) $\leq$ n-1.

Then since each $g_k'(x)$ is the intersection of finitely many members of $\bigcup_{i\in\omega} \mathcal{V}_i$, we have Ind($\partial g_k'(x)$) $\leq$ n-1.

Suppose y $\in \partial G_k(H)$. Then by the proof of Claim IV, we see that there exists m $\in \omega$ such that

$$y \notin \mathrm{Cl}(\bigcup\{g_k'(x) : x \in H \text{ and } g_k'(x) \notin \bigcup_{i<m} \mathcal{V}_i\}).$$

Thus there is a neighborhood of y meeting only finitely many elements of $\{g_k'(x) : x \in H\}$, and so we have loc Ind($\partial G_k(H)$) $\leq$ n-1. Since X is hereditarily paracompact, Ind($\partial G_k(H)$) $\leq$ n-1.

6. Closed Images

In this section we prove Theorem 3.2. We present the main part of the proof as a series of lemmas, some of which may be of independent interest.

A map f: X $\to$ Y is *irreducible* if no proper closed subset maps onto Y.

Lemma 6.1. *Suppose* X *is stratifiable, and* $f: X \to Y$ *is a closed continuous surjection. Then there exists a closed set* $X_o \subset X$ *such that* $f|_{X_o}: X_o \to f(X_o)$ *is irreducible, and* $Y - f(X_o)$ *is open and* σ*-discrete.*

Proof. By a theorem of Okuyama [O], $Y = Y_o \cup Y_1$, where each point of Y_o has a compact pre-image, and Y_1 is σ-discrete. Let $\mathcal{E} = \{X' \subset X: X'$ is closed and $f(X') \supset Y_o\}$. Partially order $\mathcal{E}$ by inclusion. It is easy to see from the fact that $f^{-1}(y)$ is compact for each $y \in Y_o$ that every chain $\mathcal{C}$ in $\mathcal{E}$ has a lower bound, namely $\cap\mathcal{C}$. Thus $\mathcal{E}$ has a minimal element X_o. X_o is closed, and $Y - f(X_o) \subset Y_1$, hence is σ-discrete. The minimality of X_o implies that $f|_{X_o}: X_o \to f(X_o)$ is irreducible.

The next lemma is essentially due to Borges and Lutzer [BL].

Lemma 6.2. *If each closed subset of* X *has a* σ*-closure-preserving outer base, and* $f: X \to Y$ *is closed and irreducible, then each closed subset of* Y *has a* σ*-closure-preserving outer base.*

Proof. Suppose $K \subset Y$ is closed. Let $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ be an outer base for $f^{-1}(K)$ such that each $\mathcal{U}_n$ is closure-preserving. For $A \subset X$, let $f^{\#}(A) = \{y \in Y: f^{-1}(y) \subset A\}$. By [BL, Lemma 3.3], $\mathcal{U}_n^{\#} = \{f^{\#}(U): U \in \mathcal{U}_n\}$ is closure-preserving. Thus $\mathcal{U}^{\#} = \bigcup_{n\in\omega} \mathcal{U}_n^{\#}$ is a σ-closure-preserving outer base for K.

Lemma 6.3. *A closed set* $K \subset X$ *has a* σ*-closure-preserving outer base if and only if for each closed* $H \subset X - K$, *there exists a sequence* $\{G_n(H,K)\}_{n\in\omega}$ *such that*

(1) each $G_n(H,K)$ *is a regular open set containing* H*;*

(2) for each $n \in \omega$*,* $\{G_n(H,K)\colon H$ *closed,* $H \cap K = \emptyset\}$ *is interior-preserving; and*

(3) for each closed $H \subset X - K$*, there exists* $n \in \omega$ *such that* $\overline{G_n(H,K)} \cap K = \emptyset$.

Proof. To see the "if" part, suppose we are given $G_n(H,K)$'s satisfying (1)-(3). Let $\mathcal{U}_n = \{(X - G_n(H,K))^{\circ}\colon \overline{G_n(H,K)} \cap K = \emptyset\}$. By (3), $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is an outer base for K. By (1), (2), and Lemma 5.1, each $\mathcal{U}_n$ is closure-preserving.

To see the "only if" part, suppose $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is an outer base for K, where each $\mathcal{U}_n$ is closure-preserving. Define $G_n(H,K) = X - \bigcup\{\overline{U}\colon U \in \mathcal{U}_n$ and $\overline{U} \cap H = \emptyset\}$. Clearly $G_n(H,K)$ is open and contains H. Since $\mathcal{U}$ is an outer base for K, (3) holds. Since the set of complements of a closure-preserving collection is interior-preserving, (2) holds. It remains to prove that $G_n(H,K)$ is regular open. Suppose $x \notin G_n(H,K)$. Then there is $U \in \mathcal{U}_n$ with $x \in \overline{U}$ and $\overline{U} \cap H = \emptyset$. Then $U \cap \overline{G_n(H,K)} = \emptyset$, and every open set containing x meets U. Thus $x \notin \overline{G_n(H,K)}^{\circ}$.

Lemma 6.4. *Suppose* $Y = Y_0 \cup Y_1$*, where* Y *is monotonically normal,* Y_0 *is closed, and* $Y_1 = Y - Y_0$. *Let* K *be closed in* Y. *If* $\mathcal{U}$ *is an interior-preserving collection of relatively open subsets of* Y_0 *whose closures miss* K*, then one can assign to each* $U \in \mathcal{U}$ *a set* U^* *open in* Y *such that*

(1) $U^* \cap Y_o = U$;

(2) $\overline{U^*} \cap Y_o = \overline{U}$;

(3) $\overline{U^*} \cap K = \emptyset$; *and*

(4) if $y \in \cap \mathcal{U}'$, *where* $\mathcal{U}' \subset \mathcal{U}$, *then* $y \in [\cap\{U^*: U \in \mathcal{U}'\}]^o$.

Proof. According to [B_2, Theorem 2.4], since Y is monotonically normal, for each $x \in Y$ and open neighborhood U of x, one can assign an open neighborhood U_x of x such that

(i) $U \subset V \Rightarrow U_x \subset V_x$;

(ii) $U_x \cap V_y \neq \emptyset \Rightarrow x \in U$ or $y \in V$.

Note that (ii) implies $U_x \subset U$.

For each $U \in \mathcal{U}$, let $U^* = \bigcup_{x \in U} [(U \cup Y_1) - K]_x$. That (1) is satisfied is obvious. To see (2), suppose $y \in (\overline{U^*} \cap Y_o) - \overline{U}$. Let W be an open neighborhood of y such that $W \cap U \neq \emptyset$. Since $W_y \cap U^* \neq \emptyset$, there exists $x \in U$ such that $W_y \cap [(U \cup Y_1) - K)_x \neq \emptyset$. This contradicts property (ii) above. To see (4), suppose $y \in \cap\mathcal{U}'$, where $\mathcal{U}' \subset \mathcal{U}$. Then $[((\cap\mathcal{U}') \cup Y_1) - K]_y \subset [(U \cup Y_1) - K]_y \subset U^*$ for each $U \in \mathcal{U}'$.

It remains to prove (3). Suppose $y \in \overline{U^*} \cap K$. Since $\overline{U} \cap K = \emptyset$, it follows from (2) that $y \in Y_1$. Then $(Y_1)_y \cap U^* \neq \emptyset$, so there exists $x \in U$ such that $(Y_1)_y \cap [U \cup Y_1) - K]_x \neq \emptyset$, again contradicting (ii).

Lemma 6.5. *Let* X *be stratifiable and* σ*-discrete, and suppose for each* $x \in X$ *we have assigned a neighborhood* O(x) *of* x. *Then one can assign to each* $x \in X$ *an open neighborhood* U(x) *of* x *such that*

(1) $U(x) \subset O(x)$;

(2) $y \in U(x) \Rightarrow U(y) \subset U(x)$;

(3) if $H \subset X$ *is closed, then* $\bigcup_{x \in H} U(x)$ *is open and closed.*

Proof. The proof is similar to that of [G_2, Theorem 1]. Let $X = \bigcup_{n \in \omega} F_n$, where each F_n is closed discrete, and $F_m \cap F_n = \emptyset$ if $m \neq n$. For each $x \in X$, let $n(x)$ be the least integer such that $x \in F_{n(x)}$. Let D be a monotone normality operator for X. Inductively, define, for each $x \in X$, a set $U(x)$ containing x such that

(i) $\{U(x): x \in F_n\}$ is a discrete collection of open and closed sets; and

(ii) $U(x) \subset O(x) \cap D(\{x\}, \bigcup_{i<n(x)} F_i) \cap (\cap\{U(y): x \in U(y)$ and $n(y) < n(x)\}$.

Since X is 0-dimensional and collectionwise-normal, the above construction can easily be carried out. These $U(x)$'s clearly satisfy (1). Also, if $y \in U(x)$, then $n(x) < n(y)$, so $U(y) \subset U(x)$ by (ii). Thus (2) holds. Finally, to see (3), suppose H is closed and $y \notin \bigcup_{x \in H} U(x)$. Then $D(H,\{y\}) \supset D(\{x\}, \bigcup_{i<n(x)} F_i)$ for all $x \in H$ with $n(x) > n(y)$. Thus $D(H,\{y\}) \supset \cup\{U(x): x \in H, n(x) > n(y)\}$, and so $y \notin \overline{\cup\{U(x): x \in H, n(x) > n(y)\}}$. But $\cup\{U(x): x \in H, n(x) \leq n(y)\}$ is open and closed. Thus $y \notin \overline{\bigcup_{x \in H} U(x)}$.

Lemma 6.6. *Suppose* Y *is stratifiable,* $Y = Y_0 \cup Y_1$, *where* Y_0 *is closed,* Y_1 *is* σ*-discrete, and* $Y_0 \cap Y_1 = \emptyset$. *If every closed subset of* Y *has a* σ*-closure-preserving outer* *base in* Y_o, *then every closed subset of* Y *has a* σ-*closure-preserving outer base.*

Proof. For a closed set $H \subset Y$, let $H_o = H \cap Y_o$ and $H_1 = H \cap Y_1$. Let $K \subset Y$ be closed. We will show that K has a σ-closure-preserving outer base.

For closed $H \subset Y - K$, let $G_n(H_o,K_o)$ be a relatively open subset of Y_o having the properties of Lemma 6.3. If $\overline{G_n(H_o,K_o)} \cap K = \emptyset$, let $G_n(H_o,K_o)^*$ be as in Lemma 6.4. Otherwise, let $G_n(H_o,K_o)^* = Y$.

Let $\{g_n(y): y \in Y, n \in \omega\}$ have the properties of Theorem 2.1. Let $Y_o = \bigcap_{n\in\omega} U_n$, where each U_n is open and contains $\overline{U}_{n+1}$. For each $x \in Y_1 - K$, let $O(x) = [(U_{n(x)} - \overline{U}_{n(x)+2}) \cap g_{n(x)}(x)] - K$, where $n(x)$ is the largest integer such that $x \in U_{n(x)}$. Let $U(x)$ be as in Lemma 6.5 applied to Y_i.

Now for H closed, $H \subset Y - K$, and $n \in \omega$, define

$$V_n(H,K) = G_n(H_o,K_o)^* \cup (\cup\{U(x): x \in [G_n(H_o,K_o)^* \cup H] \cap Y_1\}).$$

We will show that properties (1)-(3) of Lemma 6.3 hold for the $V_n(H,K)$'s.

First we will show that $\overline{V_n(H,K)} \cap Y_o = \overline{G_n(H_o,K_o)}$. Suppose not. Then there exists $y \in V_n(H,K) - \overline{G_n(H_o,K_o)}$. Observe by Lemma 6.5 that $V_n(H,K) \cap Y_1$ is open and closed in Y_1. Thus $y \in Y_o$. By Lemma 6.4, $y \notin G_n(H_o,K_o)^*$. Also $y \notin H$, so there exists $n_o \in \omega$ such that $y \notin Cl(\cup\{g_{n_o}(x): x \in G_n\overline{(H_o,K_o)^*} \cup H\})$. Thus $y \notin Cl(\cup\{U(x): x \in \overline{(G_n(H_o,K_o)^*} \cup H) \cap Y_1 \cap U_{n_o}\})$. But $Cl(\cup\{U(x): x \in (G_n(H_o,K_o)^* \cup H) \cap Y_1 \cap (Y - U_{n_o})\}) \subset Y - U_{n_o+1}$. Thus $y \notin \overline{V_n(H,K)}$, contradiction.

Clearly $V_n(H,K)$ is open and contains H. Suppose $y \in \overline{V_n(H,K)} - V_n(H,K)$. Since $V_n(H,K) \cap Y_1$ is closed in Y_1, we have $y \in Y_o \cap \overline{V_n(H,K)} = \overline{G_n(H_o,K_o)}$. Since $G_n(H_o,K_o)$ is regular open in Y_o, each neighborhood of Y contains a point $z \in Y_o - \overline{G_n(H_o,K_o)}$. Then $z \notin \overline{V_n(H,K)}$. Thus $V_n(H,K)$ is regular open, and so property (1) of Lemma 6.3 holds.

To see (2), suppose $\mathcal{H}'$ is a collection of closed sets missing K, and $y \in \bigcap_{H \in \mathcal{H}'} V_n(H,K)$. We may assume $V_n(H,K) \neq Y$. Then $y \notin K$. If $y \in Y_1$, then $U(y) \subset V_n(H;K)$ for each $H \in \mathcal{H}'$. If $y \in Y_o$, then $y \in \bigcap_{H \in \mathcal{H}'} G_n(H_o,K_o)$, so by Lemma 6.4, $y \in (\bigcap_{H \in \mathcal{H}'} G_n(H_o,K_o)^*)^o \subset \bigcap_{H \in \mathcal{H}'} V_n(H,K)$. Thus (2) holds.

Finally, to see (3), let $H \subset X - K$ be closed. There exists n such that $\overline{G_n(H_o,K_o)} \cap K_o = \emptyset$. If $y \in \overline{V_n(H,K)} \cap K$, then since $\overline{V_n(H,K)} \cap Y_o = \overline{G_n(H_o,K_o)}$, we have $y \in Y_1$. But $V_n(H,K) \cap Y_1$ is closed in Y_1 and misses K. Thus $\overline{V_n(H,K)} \cap K = \emptyset$.

Proof of Theorem 3.2. Let us call the property of Theorem 3.2 property (*). Suppose X is a stratifiable space satisfying (*), and let $f: X \to Y$ be a closed map of X onto Y. Then Y is stratifiable. We need to show that Y satisfies (*).

Let $K \subset Y$ be closed. Since $f|_{f^{-1}(K)}$ is closed, by Lemma 6.1, there exists a closed set $K_o \subset K$ such that K_o is the closed irreducible image of a closed subset of $f^{-1}(K)$, and $K - K_o$ is σ-discrete. By Lemma 6.2, and the fact that every closed subset of X satisfies (*), we see that every closed subset of K_o has a σ-closure-preserving outer base in K_o. Then by Lemma 6.6, every closed subset of K has a σ-closure-preserving outer base in K. Thus Y satisfies (*).

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