EXTENSIONS OF CONTINUOUS INCREASING FUNCTIONS

by

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1. Introduction

There is an extensive mathematical literature devoted to the problem of extending a continuous function to a larger space. In this note we consider the problem of extending continuous, increasing functions. We consider functions between two ordered topological spaces in the sense of Nachbin [4]. Nachbin [4] proves a theorem concerning extensions of continuous, increasing functions which is analogous to the Tietze extension theorem. He also constructs a compactification of an ordered topological space which is analogous to the Stone-Cech compactification of a topological space and characterizes this compactification by an extension property involving continuous, increasing functions. Our main result is a generalization of a theorem due to Taimanov [7]. We apply this result to ordered topological spaces which are determined by quasi-proximities. We show that the qp-continuous functions are the only continuous, increasing functions from such a space into a compact ordered topological space that have a continuous, increasing extension to the associated order compactification [1], [3]. This is a generalization of a theorem of Smirnov [6, Theorem 12]. The author is indebted to W. F. Lindgren for suggestions leading to an improvement
of an earlier version of this article. For further information on ordered topological spaces and quasi-topologies, the reader is referred to [2] and [4].

2. The Main Result

An ordered topological space is a triple $(X,J,\leq)$ where $(X,J)$ is a topological space, $\leq$ is a partial order on $X$, and $\{(x,y): x \leq y\}$ is closed in $X \times X$. A subset $S$ of $X$ is said to be increasing if $x \in S$ whenever $x \geq a$ for some $a \in S$. $S$ is decreasing if $X - S$ is increasing. Let $A \subseteq X$. $\text{Cl}_i A = \cap \{H: H$ is closed, increasing, $A \subseteq H\}$, $\text{Cl}_d A$ is defined analogously. $\text{Cl}_i$ and $\text{Cl}_d$ both define Kuratowski closure operators on $X$.

Theorem (2.1). Let $(X,J,\leq)$ be an ordered topological space, and let $D$ be a dense subset of $X$. Let $Y$ be a compact ordered topological space, and let $f: D \to Y$ be continuous and increasing. Then $f$ has a continuous, increasing extension $F: X \to Y$ if and only if $\text{Cl}_i f^{-1}(A) \cap \text{Cl}_d f^{-1}(B) = \emptyset$ whenever $\text{Cl}_i A \cap \text{Cl}_d B = \emptyset$, where the former closures are taken in $X$.

Proof. Let $F: X \to Y$ be a continuous, increasing extension of $f$. Suppose that $A$ and $B$ are subsets of $Y$ and $\text{Cl}_i A \cap \text{Cl}_d B = \emptyset$. From

$\text{Cl}_i F^{-1}(A) \subseteq F^{-1}(\text{Cl}_i A)$, and

$\text{Cl}_d F^{-1}(B) \subseteq F^{-1}(\text{Cl}_d B)$,

it follows that $\text{Cl}_i f^{-1}(A) \cap \text{Cl}_d f^{-1}(B) = \emptyset$. 
Conversely, let \( x \in X \) and let \( \mathcal{N}_x \) be the collection of all neighborhoods of \( x \). Let \( \mathcal{J}(x) = \{ \text{Cl}_i f(D \cap W) : W \in \mathcal{N}_x \} \) \( \cup \{ \text{Cl}_d f(D \cap W) : W \in \mathcal{N}_x \} \). Since \( Y \) is compact, \( \cap \mathcal{J}(x) \neq \emptyset \).

We now show that \( \cap \mathcal{J}(x) \) is a singleton. Suppose \( y_1, y_2 \in \cap \mathcal{J}(x) \), \( y_1 \neq y_2 \). Without loss of generality we may assume that \( \text{Cl}_d \{ y_1 \} \cap \text{Cl}_i \{ y_2 \} = \emptyset \). Since \( Y \) is a normally ordered space, there exists an open decreasing set \( G_1 \) and an open increasing set \( G_2 \) such that \( \text{Cl}_d \{ y_1 \} \subseteq G_1 \), \( \text{Cl}_i \{ y_2 \} \subseteq G_2 \), and \( \text{Cl}_d G_1 \cap \text{Cl}_i G_2 = \emptyset \). Clearly, \( \text{Cl}_d f^{-1}(G_1) \cap \text{Cl}_i f^{-1}(G_2) = \emptyset \).

Put
\[
U_1 = X - \text{Cl}_d f^{-1}(G_1), \quad \text{and} \quad U_2 = X - \text{Cl}_i f^{-1}(G_2).
\]
Then \( x \in U_{k_0} \), \( k_0 = 1 \) or \( 2 \), say \( k_0 = 1 \). Clearly \( G_1 \cap f(D - \text{Cl}_d f^{-1}(G_1)) = \emptyset \). Since \( G_1 \) is open, decreasing, we have \( G_1 \cap \text{Cl}_i (f(D - \text{Cl}_d f^{-1}(G_1))) = \emptyset \). Therefore \( y_1 \notin \text{Cl}_i f(D - \text{Cl}_d f^{-1}(G_1)) \) and since \( \text{Cl}_i f(D - \text{Cl}_d f^{-1}(G_1)) = \text{Cl}_i f(D \cap U_1) \) we have \( y_1 \notin \cap \mathcal{J}(x) \), a contradiction. Define \( F(x) = \cap \mathcal{J}(x) \). Clearly \( F \) is an extension of \( f \). We shall now demonstrate that \( F \) is continuous. Let \( V \) be a neighborhood of \( F(x) \) in \( Y \). Since each set of the form \( \text{Cl}_i f(D \cap W) \) and \( \text{Cl}_d f(D \cap W) \) is compact, there exists a set \( U \in \mathcal{N}_x \) such that
\[
\text{Cl}_i f(D \cap U) \cap \text{Cl}_d f(D \cap U) \subseteq V.
\]
If \( y \in U \), then
\[
F(y) \in \text{Cl}_i f(D \cap U) \cap \text{Cl}_d f(D \cap U) \subseteq V.
\]
Hence \( F(U) \subseteq V \). It remains to show that \( F \) is increasing. Suppose \( F(x) \leq F(y) \) is false. Put \( A = \{ F(x) \} \) and \( B = \{ F(y) \} \)
Then $\text{Cl}_1 A \cap \text{Cl}_d B = \emptyset$. Since $Y$ is a normally ordered space [4] there exist an open increasing set $G$ and an open decreasing set $H$ such that $\text{Cl}_1 A \subset G$, $\text{Cl}_d B \subset H$ and $\text{Cl}_1 G \cap \text{Cl}_d H = \emptyset$. By hypothesis, $\text{Cl}_1 f^{-1}(G) \cap \text{Cl}_d f^{-1}(H) = \emptyset$.

Suppose $x \notin \text{Cl}_1 f^{-1}(G)$; then $W = X - \text{Cl}_1 f^{-1}(G) \in \mathcal{N}$ and $f(D \cap W) \cap G \neq \emptyset$ so that $F(x) \in \text{Cl}_d f(D \cap W) \subset X - G$, a contradiction. Similarly $y \in \text{Cl}_d f^{-1}(H)$ so $x < y$ is false. This completes the proof.

3. Order Compactifications

A quasi-proximity is a binary relation on $\mathcal{P}(X)$ satisfying most of the axioms used to define a proximity. The only difference is that a quasi-proximity is not assumed to be symmetric. For further information on quasi-proximities, the reader is referred to [1], [2], and [5].

A quasi-proximity $\delta$ determines an ordered topological space $(X, \mathcal{J}, \preceq)$ if the proximity $\delta \vee \delta^{-1}$ is compatible with $(X, \mathcal{J})$ and $\{x\} \delta \{y\}$ if and only if $x \preceq y$. In [1] the authors prove that a compact ordered topological space is determined by a unique quasi-proximity. This quasi-proximity is defined by: $A \delta B$ if and only if $\text{Cl}_1 A \cap \text{Cl}_d B \neq \emptyset$. For the remaining part of this paper, $\delta$ will always denote this quasi-proximity. An ordered topological space $(\tilde{X}, \tilde{\mathcal{J}}, \preceq)$ is an order compactification of $(X, \mathcal{J}, \preceq)$, if $(X, \mathcal{J})$ is a dense subspace of the compact space $(\tilde{X}, \tilde{\mathcal{J}})$, and the restriction of $\preceq$ to $X \times X$ is $\preceq$.

Theorem (3.1) [1, Theorem 5.16]. Let $X$ be an ordered topological space.
(i) $X$ has an order compactification if and only if it is determined by a quasi-proximity.

(ii) If $\delta$ determines $X$, then there is an order compactification $\tilde{X}$ of $X$ such that $\delta$ is the restriction to $X \times X$ of the quasi-proximity $\delta_o$ on $\tilde{X}$.

(iii) Two order compactifications of $X$ are equivalent if and only if they have the same associated quasi-proximity.

Proposition (3.2). Let $(X, J, \leq)$ be determined by the quasi-proximity $\delta_o$ and let $f: X \to Y$, where $Y$ is a compact ordered topological space. Then $f$ is continuous and increasing if and only if $f$ is qp-continuous.

Theorem (3.3). Let $X$ be an ordered topological space determined by a quasi-proximity $\delta$. Let $\tilde{X}$ be the order compactification corresponding to $\delta$, and let $Y$ be any compact ordered topological space. Then a continuous, increasing function $f: X \to Y$ has a continuous, increasing extension $F: \tilde{X} \to Y$ if and only if $f$ is qp-continuous.

Proof. Suppose $A, B$ are subsets of $Y$ such that $\text{Cl}_i A \cap \text{Cl}_d B = \emptyset$, then $A \delta_o B$ and since $f$ is qp-continuous we have $f^{-1}(A) \delta_o f^{-1}(B)$. Since $(X, \delta)$ is a subspace of $(\tilde{X}, \delta_o)$, we have $f^{-1}(A) \delta_o f^{-1}(B)$ and consequently $\text{Cl}_i f^{-1}(A) \cap \text{Cl}_d f^{-1}(B) = \emptyset$, where the closures are taken in $\tilde{X}$. The result now follows from Theorem 2.1.

References


Southern Illinois University

Carbondale, Illinois 62901