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FUNCTION SPACES WHICH ARE k-SPACES

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The space of continuous real-valued functions on X with the compact-open topology, denoted by $C_\kappa(X)$, is first countable (in fact metrizable) if and only if X is hemicompact [1]. We study in this paper certain properties of $C_\kappa(X)$ which are more general than first countability. In particular, Theorems 1 and 2 characterize when $C_\kappa(X)$ is a k-space and when it has countable tightness. The proofs of these theorems are similar to the proofs of analogous theorems in [2], where the function spaces have the topology of pointwise convergence, except that modifications must be made to deal with compact sets (such as using Ascoli's theorem) instead of finite sets; and for this reason we do not include the proofs. *Throughout this paper all spaces will be completely regular* T_1*-spaces.*

A collection $\mathcal{U}$ of open subsets of a space X will be called an *open cover for compact subsets of* X provided every compact subset of X is contained in some member of $\mathcal{U}$. Furthermore, if $\{\mathcal{U}_n\}$ is a sequence of such covers, then a *residual compact-covering string from* $\{\mathcal{U}_n\}$ will be a sequence $\{U_n\}$ such that each $U_n \in \mathcal{U}_n$ and for every compact subset A of X, there exists an integer N so that $A \subseteq U_n$ for each $n \geq N$.

1. *Theorem. The following are equivalent.*

(a) $C_\kappa(X)$ *is a* k*-space.*

(b) $C_K(X)$ *is a Fréchet space.*

(c) Every sequence of open covers for compact subsets of X *has a residual compact-covering string.*

2. *Theorem.* $C_K(X)$ *has countable tightness if and only if every open cover for compact subsets of* X *has a countable subcover for compact subsets of* X.

Let us call space X k-*compact* whenever $C_K(X)$ is a k-space, and call X τ-*compact* whenever $C_K(X)$ has countable tightness. We immediately obtain the following facts.

3. *Proposition. Every hemicompact space is* k-*compact.*

4. *Proposition. Every* k-*compact space is* τ-*compact.*

5. *Proposition. Every* τ-*compact space is Lindelöf.*

6. *Proposition. Every second countable space is* τ-*compact.*

Proof. Let $\mathcal{B}$ be a countable base for X which is closed under finite unions, and let $\mathcal{U}$ be an open cover for compact subsets of X. Define

$$\mathcal{B}^* = \{B \in \mathcal{B} | B \subseteq U \text{ for some } U \in \mathcal{U}\},$$

and for each $B \in \mathcal{B}^*$, let $U(B) \in \mathcal{U}$ such that $B \subseteq U(B)$. Then define $\mathcal{U}^* = \{U(B) | B \in \mathcal{B}^*\}$, which is a countable subcollection of $\mathcal{U}$. To see that $\mathcal{U}^*$ is a cover for compact subsets of X, let A be a compact subset of X. Since $\mathcal{U}$ is a cover for compact subsets of X, there exists a $U \in \mathcal{U}$ such that $A \subseteq U$. Now for each $a \in A$, there is a $B(a) \in \mathcal{B}$ such that

$a \in B(a) \subseteq U$. Since A is compact, there exist $a_1, \cdots, a_n \in A$ such that $A \subseteq B(a_1) \cup \cdots \cup B(a_n)$. Define $B = B(a_1) \cup \cdots \cup B(a_n)$, which is in $\mathcal{B}$. Since $B \subseteq U$, then $B \in \mathcal{B}^*$. Also $A \subseteq U(B)$, so that $\mathcal{U}^*$ is indeed a cover for compact subsets of X.

7. *Proposition. Every first countable* k-*compact space is locally compact.*

Proof. Suppose that X is not locally compact at x, and let $\{U_n\}$ be a countable base at x. For every positive integer n and compact subset A of X, let U(n,A) be an open subset of X such that $\{x\} \cup A \subseteq U(n,A)$ and $U_n \setminus U(n,A) \neq \emptyset$. Then for every n, let

$$\mathcal{U}_n = \{U(n,A) | A \text{ is a compact subset of } X\},$$

which is an open cover for compact subsets of X.

Let $\{U(n,A_n)\}$ be any string from $\{\mathcal{U}_n\}$. For every n, let $a_n \in U_n \setminus U(n,A_n)$. Then $\{a_n\}$ converges to x. Let $A = \{x\} \cup \{a_n\}$, which is a compact subset of X. Then for every n, $A \not\subseteq U(n,A_n)$, so that $\{U(n,A_n)\}$ cannot be a residual compact-covering string, and thus X is not k-compact.

Since every locally compact Lindelöf space is hemicompact, we then have the following.

8. *Corollary. Every first countable* k-*compact space is hemicompact.*

Also if X is a hemicompact k-space, then $C_\kappa(X)$ is completely metrizable [3].

9. *Corollary. If* X *is first countable, then the following are equivalent.*

(a) $C_\kappa(X)$ *is a* k-*space.*

(b) $C_\kappa(X)$ *is completely metrizable.*

(c) X *is hemicompact.*

10. *Corollary. If* X *is locally compact, then the following are equivalent.*

(a) $C_\kappa(X)$ *is a* k-*space.*

(b) $C_\kappa(X)$ *is completely metrizable.*

(c) $C_\kappa(X)$ *has countable tightness.*

(d) X *is hemicompact.*

A natural question is whether X being "first countable" in Corollary 9 can be replaced by X being a "k-space." This will be true if the following question has an affirmative answer.

11. *Question.* Is every k-compact k-space, hemicompact?

Let us look finally at some examples which illustrate that the converses of the above propositions are not true. The first example follows from Propositions 6 and 7.

12. *Example.* The space of rational numbers is a τ-compact space which is not k-compact.

Also from Example 17 in [2] we obtain the following.

13. *Example.* Let F be the "Fortissimo space," which is an uncountable space with only one non-isolated point

whose neighborhoods have countable complements. Then F is k-compact but not hemicompact.

14. *Example. The Sorgenfrey line,* S, *is not* τ*-compact.*

Proof. For each compact subset A of S, define an open subset U(A) of S as follows. First let $A^* = A \cup \{0\}$, and let $a_1 = \min A^*$. If $a_1 = 0$, define $U(A) = [0,\infty)$; and we are through. Otherwise, if $a_1 \neq 0$, let $x = \max(A^* \cap [0,-a_1))$, let $b_1 = \frac{1}{2}(x - \max(A^* \cap [a_1,-x)))$, and let $a_2 = \min(A^* \cap [-x,0])$. Suppose we have gone through the n^{th} stage of this argument and found $\{a_1,\cdots,a_{n+1}\}$ and $\{b_1,\cdots,b_n\}$. Then if $a_{n+1} = 0$, define

$$U(A) = [a_1,-b_1) \cup \cdots \cup [a_n,-b_n) \cup [0,b_n) \cup [-a_n,b_{n-1}) \cup \cdots \cup [-a_2,b_1) \cup [-a_1,\infty);$$

and we are through. Otherwise continue by finding b_{n+1} and a_{n+2} as above. This process must terminate after a finite number of stages, since otherwise $\{a_n\}$ would be a strictly increasing sequence from A, contradicting the compactness of A. Therefore U(A) is well-defined.

Define $\mathcal{U} = \{U(A) | A \text{ is a compact subset of } S\}$. By construction, $A \subseteq U(A)$ for each A, so that $\mathcal{U}$ is an open cover for compact subsets of S. But each member of $\mathcal{U}$ contains only finitely many doubleton subsets of S of the form $\{x,-x\}$. Therefore $\mathcal{U}$ has no countable subcover for compact subsets of S.

We end by comparing $C_\kappa(X)$ with $C_\pi(X)$, where $C_\pi(X)$ has the topology of pointwise convergence. Whenever $C_\kappa(X)$ is first countable, then X is hemicompact and thus σ-compact. Then Proposition 6 of [2] tells us that when X is σ-compact, $C_\pi(X)$ has countable tightness. One might wonder whether $C_\kappa(X)$ has countable tightness whenever X is σ-compact, or in fact whether $C_\kappa(X)$ has countable tightness whenever $C_\pi(X)$ is first countable (equivalently, X is countable). Our final example shows that neither is true.

15. *Example. There exists a countable space* Z *which is not* τ*-compact.*

Proof. Let N be the set of natural numbers, let Q be the space of rational numbers with the usual topology, and let

$$A = \{0\} \cup (\cup\{N^n | n \in N\}).$$

Choose $\{Q_\alpha | \alpha \in A\}$ to be a pairwise disjoint family of dense subspaces of Q such that $\cup\{Q_\alpha | \alpha \in A\} = Q\backslash\{0\}$. For each $\alpha \in A$, let $\phi_\alpha \colon Q_\alpha \to N$ be a bijection. Define $\phi\colon Q \to A$ as follows:

$$\phi(0) = 0;$$

$$\phi(q) = \langle \phi_0(q) \rangle \text{ if } q \in Q_0; \text{ and}$$

$$\phi(q) = \langle i_1, \cdots, i_n, \phi_\alpha(q) \rangle \text{ if } q \in Q_\alpha \text{ for } \alpha = \langle i_1, \cdots, i_n \rangle.$$

Let $\mathcal{S} = \{\{q_0, q_1, \cdots\} \subseteq Q | q_0 = 0,\ q_{n+1} \in Q_{\phi(q_n)}$ for $n \geq 0$, and $\{q_0, q_1, \cdots\}$ converges to 0 in Q$\}$.

Now define Z = Q with the following topology. A subset $U \subseteq Z$ is open if and only if whenever $0 \in U$ then every element of $\mathcal{S}$ is eventually in U. Clearly every usual open subset of Q is open in Z. Also each point of Z different than 0 is isolated, so that Z is a 0-dimensional Hausdorff space.

Let $\mathcal{K}$ be the set of all nonempty compact subsets of Z. Note that $\mathcal{S} \subseteq \mathcal{K}$, and that if $K \in \mathcal{K}$, then $K \cap Q_\alpha$ is finite for each $\alpha \in A$. To see that the latter is true, suppose $K \cap Q_\alpha$ were infinite for some α; then $\{Z\backslash Q_\alpha\} \cup \{\{q\} | q \in K \cap Q_\alpha\}$ would be an open cover of K having no finite subcover.

For every $K \in \mathcal{K}$, define $\mathcal{S}(K) = \{\sigma \in \mathcal{S} | \sigma \not\subseteq K\}$. Also for every $\sigma \in \mathcal{S}(K)$, let $q(\sigma)$ be the first element of σ which is not in K. Finally for every $K \in \mathcal{K}$, define U(K) as follows. If $0 \notin K$, then take $U(K) = K$, which is a finite open subset of Z. If $0 \in K$, define

$$U(K) = Z\backslash\{q(\sigma) | \sigma \in \mathcal{S}(K)\},$$

which certainly contains K.

To see that U(K) is open in Z, let $\sigma \in \mathcal{S}$. We wish to show that σ is eventually in U(K). We may suppose that $\sigma \in \mathcal{S}(K)$, say $\sigma = \{q_0, q_1, \cdots\}$. Then there exists a $k \geq 1$ such that $q(\sigma) = q_k$. Now let $n > k$, and take any $\bar{\sigma} = \{\bar{q}_0, \bar{q}_1, \cdots\} \in \mathcal{S}(K)$. If $q(\bar{\sigma})$ were to equal q_n, then $\bar{q}_n = q_n$, which implies $\bar{q}_{n-1} = q_{n-1}, \cdots, \bar{q}_k = q_k$. But this contradicts $q(\bar{\sigma}) = q_n$ since $q_k \notin K$. Therefore $q_n \notin \{q(\bar{\sigma}) | \bar{\sigma} \in \mathcal{S}(K)\}$, so that $q_n \in U(K)$. Hence σ is eventually in U(K), so that U(K) is open in Z.

Now define $\mathcal{U} = \{U(K) | K \in \mathcal{K}\}$, which is an open cover for compact subsets of Z. To see that no countable subfamily of $\mathcal{U}$ is a cover for compact subsets of Z, let

$\{K_m | m \in N\} \subseteq \mathcal{K}$. Define $K \in \mathcal{K}$ as follows. First let $j_0 = 0$ and $q_0 = 0$. Suppose integers $j_0 < j_1 < \cdots < j_{n-1}$ and elements $q_0, q_1, \cdots, q_{n-1}$ from Z have been defined so that for each $0 < i < n$,

$$q_i \in Q_{\sigma(q_{i-1})} \backslash (U(K_{j_{i-1}}) \cup \cdots \cup U(K_{j_i - 1})).$$

If for every m, $\{q_0, \cdots, q_{n-1}\} \not\subseteq U(K_m)$, then define $K = \{q_0, \cdots, q_{n-1}\}$. Otherwise we continue and choose i_n to be the first m such that $\{q_0, \cdots, q_{n-1}\} \subseteq U(K_m)$. Now $U(K_{i_n}) \cap Q_{\phi(q_{n-1})} = K_{i_n} \cap Q_{\phi(q_{n-1})}$, which is finite. Since $Q_{\phi(q_{n-1})}$ is dense in Q, there exists a $q_n \in Q_{\phi(q_{n-1})} \backslash U(K_{i_n})$ such that $|q_n| < \frac{1}{n}$.

Then by induction, we have either defined K as a finite subset of Z, or we have defined the sequence $\{q_0, q_1, \cdots\} \in \mathcal{S}$. In the latter case, define $K = \{q_0, q_1, \cdots\}$, so that in either case $K \in \mathcal{K}$. Also by construction, $K \not\subseteq U(K_m)$ for any m, so that $\mathcal{U}$ has no countable subcover for compact subsets of Z.

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