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by

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This contribution is a continuation of the second author's note [3]. In contrast to [3], here the strategic situation of Player II in the game $G(X,Y)$ is considered (definitions below). Assuming Y is a separable metric space, the following is shown: (a) $\Rightarrow$ (b) $\Rightarrow$ (c), where

(a) $Y - X$ contains an analytic set which is not Borel separated from X,

(b) Player II has a winning strategy in $G(X,Y)$, and

(c) $Y - X$ contains a copy of the Cantor discontinuum.

Finally, some corollaries are derived and some open questions stated.

We recall the definition of the game $G(X,Y)$ of [3]. Let X be a subset of a topological space Y. Player I chooses a sequence $\underline{E}_1 = \langle E(1,1), E(1,2), \cdots \rangle$ of subsets of X so that $\cup \underline{E}_1 = X$. Then Player II chooses $k_1 \in N$. Assume inductively that $\underline{E}_1, k_1, \cdots, \underline{E}_n, k_n$ have been chosen. Then Player I chooses a sequence $\underline{E}_{n+1} = \langle E(n+1,1), E(n+1,2), \cdots \rangle$ of subsets of X so that $\cup \underline{E}_{n+1} = E(n,k_n)$. After this Player II chooses $k_{n+1} \in N$. Player I wins the play $\langle \underline{E}_1, k_1, \underline{E}_2, k_2, \cdots \rangle$ of $G(X,Y)$ if $\cap \{\overline{E(n,k_n)} : n \in N\} \subset X$, otherwise Player II wins.

A subset X of a topological space Y is said to be a Souslin set in Y (more precisely: a Souslin-F set in Y) if there is an indexed family

$$\{F(k_1,\cdots,k_n): \langle k_1,\cdots,k_n\rangle \in N^n,\ n \in N\}$$

of closed subsets of Y so that

$$X = \bigcup\{\bigcap\{F(k_1,\cdots,k_n): n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\}.$$

Theorem 1 [3]. *Player I has a winning strategy in* G(X,Y) *iff* X *is a Souslin set in* Y.

A subset X of a separable metric space Y is said to be analytic if either X = ∅ or there is a continuous map from the space N^N onto X.

Since Souslin and analytic subsets of Polish spaces coincide ([1], p. 482), we have from Theorem 1

Corollary 1. *Let* X *be a subset of a Polish space* Y. *Player I has a winning strategy in* G(X,Y) *iff* X *is analytic.*

Two subsets X and Z of a separable metric space Y are said to be Borel separated if there is a Borel set B in Y such that $X \subset B$ and $Z \subset Y - B$.

Theorem 2. *If* X *is a subset of a separable metric space* Y *and* Y - X *contains an analytic set* Z *which is not Borel speareted from* X, *then Player II has a winning strategy in* G(X,Y). *In particular, if* Y - X *is analytic non-Borel, then Player II has a winning strategy in* G(X,Y).

By Corollary 1 and Theorem 2 we obtain

Corollary 2. *If* X *is analytic or co-analytic in a Polish space* Y, *then the game* G(X,Y) *is determined.*

Question 1. Let X belong to the σ-algebra generated

generated by analytic subsets of an uncountable Polish space Y. Is then G(X,Y) determined?

Theorem 3. *If* X *is a subset of a separable metric space* Y *and Player II has a winning strategy in* G(X,Y), *then* Y - X *contains a copy of the Cantor discontinuum.*

Question 2. Let X be a Lusin set on the real line R (i.e., X is uncountable and $X \cap F$ is at most countable whenever F is nowhere dense in R). Does Player II have a winning strategy in G(X,R)?

A subset X of an uncountable Polish space Y is said to be a Bernstein set if neither X nor Y - X contains a copy of the Cantor discontinuum. Each uncountable Polish space contains a Bernstein set ([1], p. 514). Hence by Corollary 1 and Theorem 3 we obtain

Corollary 3. *If* X *is a Bernstein set in an uncountable Polish space* Y, *then the game* G(X,Y) *is undetermined.*

Major question: Is the sufficient condition for the existence of a winning strategy for Player II given in Theorem 2 also necessary? Indeed, granted a winning strategy t for Player II, we are unable to determine the descriptive character of the set of points arising as outcomes of arbitrary plays in G(X,Y) with II playing according to t.

Proofs. The proof of Theorem 1 given below differs from that of [3] and, moreover, is much simpler.

Proof of Theorem 1. Let s be a strategy of Player I in G(X,Y). Then s determines a Souslin set

$$X_s = \cup\{\cap\{\overline{E_s(k_1,\cdots,k_n)}: n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\}$$

in Y as follows: $E_s(k_1) = s(\emptyset)(k_1)$, $E_s(k_1,k_2) = s(k_1)(k_2)$, $E_s(k_1,k_2,k_3) = s(k_1,k_2)(k_3)$, and so on. It is easy to check that $X_s \supset X$. Clearly, s is a winning strategy iff $X_s = X$. Hence, in particular, if s is a winning strategy of Player I, then X is a Souslin set in Y. To prove the converse implication, assume that X is a Souslin set in Y, i.e.,

$$X = \cup\{\cap\{F(k_1,\cdots,k_n): n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\},$$

where each $F(k_1,\cdots,k_n)$ is closed in Y. Let us put $E(k_1,\cdots,k_n) = \cup\{\cap\{F(j_1,\cdots,j_m): m \in N\}: \langle j_1,j_2,\cdots\rangle \in B(k_1,\cdots,k_n)\}$, where

$$B(k_1,\cdots,k_n) = \{\langle i_1,i_2,\cdots\rangle \in N^N: \langle i_1,\cdots,i_n\rangle = \langle k_1,\cdots,k_n\rangle\}.$$

It is easy to verify that for any $\langle k_1,k_2,\cdots\rangle \in N^N$

$$E(k_1,\cdots,k_n) \subset F(k_1,\cdots,k_n),$$
$$\cap\{F(k_1,\cdots,k_n): n \in N\} = \cap\{E(k_1,\cdots,k_n): n \in N\},$$
$$\cup\{E(k): k \in N\} = X, \text{ and}$$
$$\cup\{E(k_1,\cdots,k_n,k): k \in N\} = E(k_1,\cdots,k_n).$$

Hence

$$X = \cup\{\cap\{E(k_1,\cdots,k_n): n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\}$$

and a winning strategy for Player I can be defined as follows: $s(\emptyset)(k_1) = E(k_1)$, $s(k_1)(k_2) = E(k_1,k_2)$, $s(k_1,k_2)(k_3) = E(k_1,k_2,k_3)$, and so on. The proof of Theorem 1 is complete.

Lemma. Let X *and* Z *be subsets of a topological space* Y, *and let* $X = \cup\{X_m: m \in N\}$ *and* $Z = \cup\{Z_n: n \in N\}$. *If* X *and* Z *are not Borel separated, then there are* $m \in N$ *and* $n \in N$ *so that* X_m *and* Z_n *are not Borel separated.*

The lemma is classical, see [1], p. 485 or [2], p. 228, for proof (which is easy).

Proof of Theorem 2. Let us assume that Y - X contains an analytic set Z which is not Borel separated from X. Let f be a continuous map from N^N onto Z and let

$$F(j_1,\cdots,j_n) = f(B(j_1,\cdots,j_n)),$$

where, as before,

$$B(j_1,\cdots,j_n) = \{\langle i_1,i_2,\cdots\rangle \in N^N: \langle i_1,\cdots,i_n\rangle = \langle j_1,\cdots,j_n\rangle\}.$$

Then

$$\cup\{F(j): j \in N\} = Z,$$

$$\cup\{F(j_1,\cdots,j_n,j): j \in N\} = F(j_1,\cdots,j_n), \text{ and}$$

$$\text{diam } F(j_1,\cdots,j_n) \to 0 \text{ as } n \to \infty$$

for each $\langle j_1,j_2,\cdots\rangle \in N^N$. We shall define a winning strategy t for Player II in G(X,Y) as follows. Let $\underline{E}_1 = \langle E(1,1),E(1,2),\cdots\rangle$, where $\cup\underline{E}_1 = X$. Since X and Z are not Borel separated, it follows from the lemma that there is $k_1 \in N$ and $j_1 \in N$ so that $E(1,k_1)$ and $F(j_1)$ are not Borel separated. We set $t(\underline{E}_1) = k_1$. Let $\underline{E}_2 = \langle E(2,1), E(2,2),\cdots\rangle$, where $\cup\underline{E}_2 = E(1,k_1)$. Again by the lemma we infer the existence of $k_2 \in N$ and $j_2 \in N$ such that $E(2,k_2)$ and $F(j_1,j_2)$ are not Borel separated. We set $t(\underline{E}_1,\underline{E}_2) = k_2$, and so on. Since $E(n,k_n)$ and $F(j_1,\cdots,j_n)$ are not Borel separated, it follows that

$$\overline{E(n,k_n)} \cap \overline{F(j_1,\cdots,j_n)} \neq 0.$$

Since $\cap\{\overline{F(j_1,\cdots,j_n)}: n \in N\} = \{z\} \subset Z$, where $z = f(j_1,j_2,\cdots)$, and diam $F(j_1,\cdots,j_n) \to 0$ as $n \to \infty$, we also have $z \in \cap\{\overline{E(n,k_n)}: n \in N\}$. Indeed, if U is an open neighbourhood of z in Y and $n \in N$, then there is $m \geq n$ such that $\overline{F(j_1,\cdots,j_m)} \subset U$. Since $\overline{F(j_1,\cdots,j_m)} \cap \overline{E(m,k_m)} \neq 0$, we have $\overline{E(m,k_m)} \cap U \neq 0$ and so $E(n,k_n) \cap U \neq 0$, because $E(n,k_n) \supset E(m,k_m)$. Thus $z \in \overline{E(n,k_n)}$. Finally, $(Y - X) \cap \cap\{\overline{E(n,k_n)}: n \in N\} \neq 0$ and thus t is a winning strategy for Player II. The proof is complete.

Proof of Theorem 3. Let t be a fixed winning strategy for Player II in $G(X,Y)$, where Y is a separable metric space. If $\langle \underline{E}_1,k_1,\cdots,\underline{E}_n,k_n\rangle$ is a partial t-play (i.e., a partial play of $G(X,Y)$ in which Player II follows the strategy t), let $T(\underline{E}_1,\cdots,\underline{E}_n) = \underline{E}_n(k_n)$, the set determined by Player II's nth move; let $T(\underline{E}_1,\cdots,\underline{E}_n) = X$ if $n = 0$. Let $M(\underline{E}_1,\cdots,\underline{E}_n)$ be the set of all sequences $\underline{E} = \langle E(k): k \in N\rangle$ such that $\cup\underline{E} = T(\underline{E}_1,\cdots,\underline{E}_n)$; i.e., $M(\underline{E}_1,\cdots,\underline{E}_n)$ is the set of all legal moves by Player I following $\langle \underline{E}_1,k_1,\cdots,\underline{E}_n,k_n\rangle$. Clearly, it will suffice to establish the following:

Claim 1. We can associate with every finite sequence $\langle b_1,\cdots,b_n\rangle$ of 0's and 1's a sequence $\underline{E}_{b_1,\cdots,b_n}$ of subsets of X, so that the following conditions are satisfied:

(1) $\underline{E}_{b_1,\cdots,b_n} \in M(\underline{E}_{b_1},\underline{E}_{b_1,b_2},\cdots,\underline{E}_{b_1,b_2,\cdots,b_{n-1}})$;

(2) $T(\underline{E}_{b_1}, \underline{E}_{b_1,b_2}, \cdots, \underline{E}_{b_1,\cdots,b_n})$ has diameter $< 1/n$;

(3) the sets $T(\underline{E}_{b_1}, \cdots, \underline{E}_{b_1,\cdots,b_n}, \underline{E}_{b_1,\cdots,b_n,0})$ and $T(\underline{E}_{b_1}, \cdots, \underline{E}_{b_1,\cdots,b_n}, \underline{E}_{b_1,\cdots,b_n,1})$ have disjoint closures.

In fact, it will suffice to prove:

Claim 2. Let $\langle \underline{E}_1, k_1, \cdots, \underline{E}_n, k_n \rangle$ be a partial t-play, and let $\mathcal{F} = \{T(\underline{E}_1, \cdots, \underline{E}_n, \underline{E}): \underline{E} \in M(\underline{E}_1, \cdots, \underline{E}_n)\}$. Then there are sets F', F" $\in \mathcal{F}$ such that F' and F" have diameter $< 1/(n+1)$ and $\overline{F'} \cap \overline{F''} = \emptyset$.

Proof of Claim 2. Let $W = \{y \in Y$: every neighbourhood of y contains a member of $\mathcal{F}\}$. It will suffice to show that W contains at least two points. Let $\mathcal{U} = \{U \subset Y$: U is open in Y, and U contains no member of $\mathcal{F}\}$. Then $\cup\mathcal{U} = Y - W$. Since Y is a separable metric space, we can write $Y - W = \cup\{U_n: n \in N\}$, $U_n \in \mathcal{U}$. Now let $E = T(\underline{E}_1, \cdots, \underline{E}_n)$, and let $\underline{E}_{n+1} = \langle E \cap W, E \cap U_1, E \cap U_2, \cdots \rangle \in M(\underline{E}_1, \cdots, \underline{E}_n)$. Since $E \cap U_n \notin \mathcal{F}$, we must have $T(\underline{E}_1, \cdots, \underline{E}_{n+1}) = E \cap W$. Now $E \cap W \subset X$, but $\overline{E \cap W} \not\subset X$ since t is a winning strategy for Player II in G(X,Y); hence $E \cap W$ is infinite. The proof is complete.

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