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by

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This contribution is a continuation of the second author's note [3]. In contrast to [3], here the strategic situation of Player II in the game $G(X,Y)$ is considered (definitions below). Assuming Y is a separable metric space, the following is shown: (a) $\Rightarrow$ (b) $\Rightarrow$ (c), where

(a) $Y - X$ contains an analytic set which is not Borel separated from X,

(b) Player II has a winning strategy in $G(X,Y)$, and

(c) $Y - X$ contains a copy of the Cantor discontinuum. Finally, some corollaries are derived and some open questions stated.

We recall the definition of the game $G(X,Y)$ of [3]. Let X be a subset of a topological space Y. Player I chooses a sequence $\underline{E}_1 = \langle E(1,1), E(1,2), \cdots \rangle$ of subsets of X so that $\cup \underline{E}_1 = X$. Then Player II chooses $k_1 \in N$. Assume inductively that $\underline{E}_1, k_1, \cdots, \underline{E}_n, k_n$ have been chosen. Then Player I chooses a sequence $\underline{E}_{n+1} = \langle E(n+1,1), E(n+1,2), \cdots \rangle$ of subsets of X so that $\cup \underline{E}_{n+1} = E(n,k_n)$. After this Player II chooses $k_{n+1} \in N$. Player I wins the play $\langle \underline{E}_1, k_1, \underline{E}_2, k_2, \cdots \rangle$ of $G(X,Y)$ if $\cap \{\overline{E(n,k_n)} : n \in N\} \subset X$, otherwise Player II wins.

A subset X of a topological space Y is said to be a Souslin set in Y (more precisely: a Souslin-F set in Y) if there is an indexed family

$$\{F(k_1,\cdots,k_n) : \langle k_1,\cdots,k_n\rangle \in N^n,\ n \in N\}$$

of closed subsets of Y so that

$$X = \bigcup\{\bigcap\{F(k_1,\cdots,k_n) : n \in N\} : \langle k_1,k_2,\cdots\rangle \in N^N\}.$$

Theorem 1 [3]. *Player I has a winning strategy in* G(X,Y) *iff* X *is a Souslin set in* Y.

A subset X of a separable metric space Y is said to be analytic if either X = ∅ or there is a continuous map from the space N^N onto X.

Since Souslin and analytic subsets of Polish spaces coincide ([1], p. 482), we have from Theorem 1

Corollary 1. *Let* X *be a subset of a Polish space* Y. *Player I has a winning strategy in* G(X,Y) *iff* X *is analytic.*

Two subsets X and Z of a separable metric space Y are said to be Borel separated if there is a Borel set B in Y such that $X \subset B$ and $Z \subset Y - B$.

Theorem 2. *If* X *is a subset of a separable metric space* Y *and* Y - X *contains an analytic set* Z *which is not Borel speared from* X, *then Player II has a winning strategy in* G(X,Y). *In particular, if* Y - X *is analytic non-Borel, then Player II has a winning strategy in* G(X,Y).

By Corollary 1 and Theorem 2 we obtain

Corollary 2. *If* X *is analytic or co-analytic in a Polish space* Y, *then the game* G(X,Y) *is determined.*

Question 1. Let X belong to the σ-algebra generated

generated by analytic subsets of an uncountable Polish space Y. Is then $G(X,Y)$ determined?

Theorem 3. *If* X *is a subset of a separable metric space* Y *and Player II has a winning strategy in* $G(X,Y)$, *then* $Y - X$ *contains a copy of the Cantor discontinuum.*

Question 2. Let X be a Lusin set on the real line R (i.e., X is uncountable and $X \cap F$ is at most countable whenever F is nowhere dense in R). Does Player II have a winning strategy in $G(X,R)$?

A subset X of an uncountable Polish space Y is said to be a Bernstein set if neither X nor $Y - X$ contains a copy of the Cantor discontinuum. Each uncountable Polish space contains a Bernstein set ([1], p. 514). Hence by Corollary 1 and Theorem 3 we obtain

Corollary 3. *If* X *is a Bernstein set in an uncountable Polish space* Y, *then the game* $G(X,Y)$ *is undetermined.*

Major question: Is the sufficient condition for the existence of a winning strategy for Player II given in Theorem 2 also necessary? Indeed, granted a winning strategy t for Player II, we are unable to determine the descriptive character of the set of points arising as outcomes of arbitrary plays in $G(X,Y)$ with II playing according to t.

Proofs. The proof of Theorem 1 given below differs from that of [3] and, moreover, is much simpler.

Proof of Theorem 1. Let s be a strategy of Player I in $G(X,Y)$. Then s determines a Souslin set

$$X_s = \cup\{\cap\{\overline{E_s(k_1,\cdots,k_n)}: n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\}$$

in Y as follows: $E_s(k_1) = s(\emptyset)(k_1)$, $E_s(k_1,k_2) = s(k_1)(k_2)$, $E_s(k_1,k_2,k_3) = s(k_1,k_2)(k_3)$, and so on. It is easy to check that $X_s \supset X$. Clearly, s is a winning strategy iff $X_s = X$. Hence, in particular, if s is a winning strategy of Player I, then X is a Souslin set in Y. To prove the converse implication, assume that X is a Souslin set in Y, i.e.,

$$X = \cup\{\cap\{F(k_1,\cdots,k_n): n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\},$$

where each $F(k_1,\cdots,k_n)$ is closed in Y. Let us put $E(k_1,\cdots,k_n) = \cup\{\cap\{F(j_1,\cdots,j_m): m \in N\}: \langle j_1,j_2,\cdots\rangle \in B(k_1,\cdots,k_n)\}$, where

$$B(k_1,\cdots,k_n) = \{\langle i_1,i_2,\cdots\rangle \in N^N: \langle i_1,\cdots,i_n\rangle = \langle k_1,\cdots,k_n\rangle\}.$$

It is easy to verify that for any $\langle k_1,k_2,\cdots\rangle \in N^N$

$$E(k_1,\cdots,k_n) \subset F(k_1,\cdots,k_n),$$

$$\cap\{F(k_1,\cdots,k_n): n \in N\} = \cap\{E(k_1,\cdots,k_n): n \in N\},$$

$$\cup\{E(k): k \in N\} = X, \text{ and}$$

$$\cup\{E(k_1,\cdots,k_n,k): k \in N\} = E(k_1,\cdots,k_n).$$

Hence

$$X = \cup\{\cap\{E(k_1,\cdots,k_n): n \in N\}: \langle k_1,k_2,\cdots\rangle \in N^N\}$$

and a winning strategy for Player I can be defined as follows: $s(\emptyset)(k_1) = E(k_1)$, $s(k_1)(k_2) = E(k_1,k_2)$, $s(k_1,k_2)(k_3) = E(k_1,k_2,k_3)$, and so on. The proof of Theorem 1 is complete.

Lemma. Let X *and* Z *be subsets of a topological space* Y*, and let* $X = \cup\{X_m: m \in N\}$ *and* $Z = \cup\{Z_n: n \in N\}$*. If* X *and* Z *are not Borel separated, then there are* $m \in N$ *and* $n \in N$ *so that* X_m *and* Z_n *are not Borel separated.*

The lemma is classical, see [1], p. 485 or [2], p. 228, for proof (which is easy).

Proof of Theorem 2. Let us assume that $Y - X$ contains an analytic set Z which is not Borel separated from X. Let f be a continuous map from N^N onto Z and let

$$F(j_1,\cdots,j_n) = f(B(j_1,\cdots,j_n)),$$

where, as before,

$$B(j_1,\cdots,j_n) = \{\langle i_1,i_2,\cdots\rangle \in N^N: \langle i_1,\cdots,i_n\rangle = \langle j_1,\cdots,j_n\rangle\}.$$

Then

$$\cup\{F(j): j \in N\} = Z,$$

$$\cup\{F(j_1,\cdots,j_n,j): j \in N\} = F(j_1,\cdots,j_n), \text{ and}$$

$$\operatorname{diam} F(j_1,\cdots,j_n) \to 0 \text{ as } n \to \infty$$

for each $\langle j_1,j_2,\cdots\rangle \in N^N$. We shall define a winning strategy t for Player II in $G(X,Y)$ as follows. Let $\underline{E}_1 = \langle E(1,1),E(1,2),\cdots\rangle$, where $\cup\underline{E}_1 = X$. Since X and Z are not Borel separated, it follows from the lemma that there is $k_1 \in N$ and $j_1 \in N$ so that $E(1,k_1)$ and $F(j_1)$ are not Borel separated. We set $t(\underline{E}_1) = k_1$. Let $\underline{E}_2 = \langle E(2,1), E(2,2),\cdots\rangle$, where $\cup\underline{E}_2 = E(1,k_1)$. Again by the lemma we infer the existence of $k_2 \in N$ and $j_2 \in N$ such that $E(2,k_2)$ and $F(j_1,j_2)$ are not Borel separated. We set $t(\underline{E}_1,\underline{E}_2) = k_2$, and so on. Since $E(n,k_n)$ and $F(j_1,\cdots,j_n)$ are not Borel separated, it follows that

$$\overline{E(n,k_n)} \cap \overline{F(j_1,\cdots,j_n)} \neq 0.$$

Since $\cap\{\overline{F(j_1,\cdots,j_n)}: n \in N\} = \{z\} \subset Z$, where $z = f(j_1,j_2,\cdots)$, and diam $F(j_1,\cdots,j_n) \to 0$ as $n \to \infty$, we also have $z \in \cap\{\overline{E(n,k_n)}: n \in N\}$. Indeed, if U is an open neighbourhood of z in Y and $n \in N$, then there is $m \geq n$ such that $\overline{F(j_1,\cdots,j_m)} \subset U$. Since $\overline{F(j_1,\cdots,j_m)} \cap \overline{E(m,k_m)} \neq 0$, we have $\overline{E(m,k_m)} \cap U \neq 0$ and so $E(n,k_n) \cap U \neq 0$, because $E(n,k_n) \supset E(m,k_m)$. Thus $z \in \overline{E(n,k_n)}$. Finally, $(Y - X) \cap \cap\{\overline{E(n,k_n)}: n \in N\} \neq 0$ and thus t is a winning strategy for Player II. The proof is complete.

Proof of Theorem 3. Let t be a fixed winning strategy for Player II in $G(X,Y)$, where Y is a separable metric space. If $\langle \underline{E}_1,k_1,\cdots,\underline{E}_n,k_n\rangle$ is a partial t-play (i.e., a partial play of $G(X,Y)$ in which Player II follows the strategy t), let $T(\underline{E}_1,\cdots,\underline{E}_n) = \underline{E}_n(k_n)$, the set determined by Player II's nth move; let $T(\underline{E}_1,\cdots,\underline{E}_n) = X$ if $n = 0$. Let $M(\underline{E}_1,\cdots,\underline{E}_n)$ be the set of all sequences $\underline{E} = \langle E(k): k \in N\rangle$ such that $\cup\underline{E} = T(\underline{E}_1,\cdots,\underline{E}_n)$; i.e., $M(\underline{E}_1,\cdots,\underline{E}_n)$ is the set of all legal moves by Player I following $\langle \underline{E}_1,k_1,\cdots,\underline{E}_n,k_n\rangle$. Clearly, it will suffice to establish the following:

Claim 1. We can associate with every finite sequence $\langle b_1,\cdots,b_n\rangle$ of 0's and 1's a sequence $\underline{E}_{b_1,\cdots,b_n}$ of subsets of X, so that the following conditions are satisfied:

(1) $\underline{E}_{b_1,\cdots,b_n} \in M(\underline{E}_{b_1},\underline{E}_{b_1,b_2},\cdots,\underline{E}_{b_1,b_2,\cdots,b_{n-1}})$;

(2) $T(\underline{E}_{b_1}, \underline{E}_{b_1,b_2}, \cdots, \underline{E}_{b_1,\cdots,b_n})$ has diameter $< 1/n$;

(3) the sets $T(\underline{E}_{b_1}, \cdots, \underline{E}_{b_1,\cdots,b_n}, \underline{E}_{b_1,\cdots,b_n,0})$ and $T(\underline{E}_{b_1}, \cdots, \underline{E}_{b_1,\cdots,b_n}, \underline{E}_{b_1,\cdots,b_n,1})$ have disjoint closures.

In fact, it will suffice to prove:

Claim 2. Let $\langle \underline{E}_1, k_1, \cdots, \underline{E}_n, k_n \rangle$ be a partial t-play, and let $\mathcal{J} = \{T(\underline{E}_1, \cdots, \underline{E}_n, \underline{E}): \underline{E} \in M(\underline{E}_1, \cdots, \underline{E}_n)\}$. Then there are sets F', $F'' \in \mathcal{J}$ such that F' and F'' have diameter $< 1/(n+1)$ and $\overline{F'} \cap \overline{F''} = \emptyset$.

Proof of Claim 2. Let $W = \{y \in Y$: every neighbourhood of y contains a member of $\mathcal{J}\}$. It will suffice to show that W contains at least two points. Let $\mathcal{U} = \{U \subset Y$: U is open in Y, and U contains no member of $\mathcal{J}\}$. Then $\cup\mathcal{U} = Y - W$. Since Y is a separable metric space, we can write $Y - W = \cup\{U_n: n \in N\}$, $U_n \in \mathcal{U}$. Now let $E = T(\underline{E}_1, \cdots, \underline{E}_n)$, and let $\underline{E}_{n+1} = \langle E \cap W, E \cap U_1, E \cap U_2, \cdots \rangle \in M(\underline{E}_1, \cdots, \underline{E}_n)$. Since $E \cap U_n \notin \mathcal{J}$, we must have $T(\underline{E}_1, \cdots, \underline{E}_{n+1}) = E \cap W$. Now $E \cap W \subset X$, but $\overline{E \cap W} \not\subset X$ since t is a winning strategy for Player II in $G(X,Y)$; hence $E \cap W$ is infinite. The proof is complete.

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