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In [Su] Sundaresan constructed a compact T_2-space X such that if Y and Z are the results of adding one and two isolated points, respectively, to X, then $X \cong Z \not\cong Y$. ('$\cong$' denotes homeomorphism.) Thus, since each of X and Y embeds in the other, there is no Schroeder-Bernstein theorem for compact T_2-spaces and embeddings. Also, $X + X \cong X + Z \cong Y + Y$, where '+' denotes discrete union, and it follows from the well-known Banach-Stone theorem [Da] that $C(X + X,R)$ and $C(Y + Y,R)$ are isometric (denoted by '$\equiv$'). This was the focus of interest in [Su]; for if R_∞^2 is R^2 with the sup norm, then $C(X,R_\infty^2) \equiv C(X + X,R) \equiv C(Y + Y,R) \equiv C(Y,R_\infty^2)$, showing that the Banach-Stone theorem cannot be extended to arbitrary real Banach spaces.

At any rate, X has a number of interesting features, all but one of which (given X) are easy to verify. More difficult is that $X \not\cong Y$; nevertheless, the proof in [Su] is unnecessarily long and indirect, as I now show.

X is obtained by pasting together the remainders of two copies of $\beta\omega$. More precisely, let $X = \omega^* \cup (\omega \times 2)$, where $\omega^* = \beta\omega\backslash\omega$, and let $\pi: X \to \beta\omega$ be the obvious projection; the topology on X is the coarsest making π continuous and each point of $N = \omega \times 2$ isolated. Let $N_i = \omega \times \{i\}$, $i \in 2$. Intuitively, $X \not\cong Y$ *because* the extra point in Y must be added to one of the N_i's, and this 'skews' the pasting-together: the two copies of ω^* no longer line up right. (In Z, of course, we can think of one new point as extending N_0, the other N_1, so that the two copies of ω^*, being similarly 'shifted,' still line up.)

To express this idea rigorously, let $P_n = \{n\} \times 2$ for $n \in \omega$, and let $\mathcal{P} = \{P_n : n \in \omega\}$. A function $f: X \to X$ *preserves pairs* iff $f[P] \in \mathcal{P}$ for all but finitely many $P \in \mathcal{P}$, and the idea is that any embedding $h: X \to X$ must preserve pairs. Otherwise, since h is 1-1, an easy recursion suffices to produce an infinite $M \subseteq \omega$ such that $\pi \circ h$ is 1-1 on $\bigcup\{P_n : n \in M\}$. Let $H_i = M \times \{i\}$ for $i \in 2$. Then $(\mathrm{cl}_X H_i)\setminus N = (\mathrm{cl}_{\beta\omega} M)\setminus\omega \neq \emptyset$ for $i \in 2$, so $(\mathrm{cl}_X h[H_0])\setminus N = (\mathrm{cl}_X h[H_1])\setminus N \neq \emptyset$. But $(\mathrm{cl}_X h[H_i])\setminus N = (\mathrm{cl}_{\beta\omega} \pi[h[H_i]])\setminus\omega$ for $i \in 2$, $\pi[h[H_0]] \cap \pi[h[H_1]] = \emptyset$, and disjoint subsets of ω have disjoint closures in $\beta\omega$, so the sets $\mathrm{cl}_X h[H_i]$ $(i \in 2)$ must be disjoint; this is the desired contradiction.

If, now, $h: Y \leftrightarrow X$ is a homeomorphism, then $h \restriction X$ preserves pairs. Let $A = \bigcup\{P_n \in \mathcal{P} : h[P_n] \in \mathcal{P}\} \cup \omega^*$. Then clearly $|X\setminus h[A]|$ is finite and even, $|Y\setminus A|$ is finite and odd, and $h \restriction (Y\setminus A)$ is a bijection between these two sets, which is absurd. Hence $X \not\approx Y$.

References

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