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IRREDUCIBLE SPACES AND PROPERTY b_1

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1. Introduction

In an unpublished paper [8] J. Chaber introduced a topological property which he called *property* b_1. Chaber showed that this property plays an important role in the study of metacompact and θ-refinable spaces. Since these classes of spaces are irreducible, it is natural to investigate the relationship between property b_1 and irreducibility. A topological space X is *irreducible* if every open cover of X has an open refinement which is a minimal cover of X. Studies of irreducible spaces have been made by R. Arens and J. Dugundji [1], J. Boone [3,4], U. Christian [9,10], the author [17,18,19], and J. Worrell and H. Wicke [21].

In this paper we investigate property b_1 and its natural variations. In particular we show in Section 2 that property b_1 is actually stronger than the notion of weakly $\overline{\theta}$-refinable but a weaker version of property b_1 is implied by weakly $\overline{\theta}$-refinable. Also in Section 3 we show that another weaker version of property b_1 always implies irreducibility. Application of these results are given in Section 4 where several unanswered questions are solved. A number of new problems are also included.

The following notions and definitions are included for the benefit of the reader.

Notation. Let $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ be a collection of subsets of a space X. We will denote $\bigcup_{\alpha \in A} F_\alpha$ by $\cup\mathcal{F}$.

Definition 1.1. A space X is called *weakly* $\bar{\theta}$*-refinable* provided every open cover $\mathcal{G}$ of X has a refinement $\cup_{i=1}^{\infty}\mathcal{G}_i$ satisfying:

(i) each $\mathcal{G}_i = \{G(\alpha,i): \alpha \in A_i\}$ is a collection of open subsets of X,

(ii) for each $x \in X$, there exists an integer n(x) such that $0 < \text{ord}(x,\mathcal{G}_{n(x)}) < \infty$,

(iii) if $x \in X$, then $x \in G_i^*$ for only finitely many i, where $G_i^* = \cup\mathcal{G}_i$.

Naturally, a cover $\cup_{i=1}^{\infty}\mathcal{G}_i$ satisfying (i)-(iii) above is called a weak $\bar{\theta}$-cover. Spaces satisfying only (i) and (ii) are called *weakly* θ*-refinable* and were introduced by Bennett and Lutzer [2].

Definition 1.2. A space X is called θ*-refinable* if every open cover $\mathcal{G}$ of X has a refinement $\cup_{i=1}^{\infty}\mathcal{G}_i$ where each $\mathcal{G}_i$ is an open cover of X and property (ii) above is satisfied.

The following property was introduced by J. Chaber in an unpublished paper [8]. This property was shown to play an important role in the study of θ-refinable and metacompact spaces as stated in the next theorem.

Definition 1.3. A space X is said to have *property* b_1 if each open cover $\mathcal{U}$ of X can be refined by a cover $\mathcal{F} = \cup_{i=1}^{\infty}\mathcal{F}_i$ such that,

$\mathcal{F}_n$ is a locally finite collection of closed sets in $X - \bigcup_{k<n} [\cup\mathcal{F}_k]$.

Theorem 1.4. *(1) A space* X *is metacompact iff* X *is almost expandable and has property* b_1.

(2) A space X *is* θ*-refinable iff* X *is almost* θ*-expandable and has property* b_1.

Properties of almost expandable and almost θ-expandable spaces are discussed in [8,13,14,16,17,20].

Definition 1.5. A collection $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ is called *hereditarily closure-preserving* (HCP) provided for every $B \subseteq A$ and every collection $\{H_\beta : \beta \in B\}$, where $H_\beta \subseteq F_\beta$, we have that $\bigcup_{\beta\in B} \overline{H}_\beta = \overline{\bigcup_{\beta\in B} H_\beta}$.

Definition 1.6. A space X is said to have *property* $B(D(\text{resp. } LF, HCP),\alpha)$ if each open cover $\mathcal{U}$ of X has a refinement $\bigcup_{s<\alpha} \mathcal{F}_s$, such that for each $s < \alpha$

(1) $\mathcal{F}_s$ is a discrete (resp. locally finite, HCP) collection of closed sets in $X - \bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$.

(2) $\bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$ is closed in X.

Remark. Note that property $B(LF,\omega_0) \equiv$ property b_1 according to Chaber [8]. It should be clear that property $B(D,\alpha) \Rightarrow$ property $B(LF,\alpha) \Rightarrow$ property $B(HCP,\alpha)$ for each α.

Definition 1.7. A collection $\mathcal{V}$ is a *"partial" refinement* of a collection $\mathcal{U}$ provided each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. (It need not be the case that $\cup\mathcal{V} = \cup\mathcal{U}$.)

2. Property B (D,ω_0) and Weakly $\bar{\theta}$-Refinable Spaces

In order to begin our study it is interesting to note that property $B(D,\omega_0)$ is stronger than the property of weak $\bar{\theta}$-refinability.

Theorem 2.1. *If a space* X *has property* $B(D,\omega_0)$ *then* X *is weakly* $\bar{\theta}$*-refinable.*

Proof. Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a refinement $\cup_{i=1}^{\infty}\mathcal{F}_i$ satisfying (1) and (2) in Definition 1.6 above. We now construct the sequence $\{\mathcal{G}_i\}_{i=1}^{\infty}$ satisfying properties (i)-(iii) of Definition 1.1 above.

Now for each $\alpha \in A$ and each $n < \omega_0$, choose $U(\alpha,n) \in \mathcal{U}$ such that $F(\alpha,n) \subseteq U(\alpha,n)$ where $F(\alpha,n) \in \mathcal{F}_n$.

Define $G(\alpha,n) = U(\alpha,n) - \bigcup_{\beta\neq\alpha} F(\beta,n) - \bigcup_{k<n} [\cup\mathcal{F}_k]$ for each $\alpha \in A$ and $n < \omega_0$ and let

$$\mathcal{G}_n = \{G(\alpha,n): \alpha \in A\}.$$

It is clear that each $\mathcal{G}_n$ is a collection of open subsets of X. Furthermore if $x \in X$ choose $n(x)$ to be the first integer for which x belongs to some member $F(\alpha,n(x))$ of $\mathcal{F}_{n(x)}$. Then x belongs to only $G(\alpha,n(x)) \in \mathcal{G}_{n(x)}$ and x belongs to no member of $\mathcal{G}_k$ for $k > n(x)$. Therefore $\cup_{i=1}^{\infty}\mathcal{G}_i$ satisfies properties (i)-(iii) in Definition 1.1 above so that X is weakly $\bar{\theta}$-refinable.

Remark. The author conjectures that property $B(D,\omega_0)$ and weakly $\bar{\theta}$-refinability are not equivalent. In fact, the author conjectures that there is a space X which is weakly $\bar{\theta}$-refinable and has property $B(D,\omega_0+1)$ but does not have property $B(D,\omega_0)$. Such examples however appear to be somewhat complicated.

Theorem 2.2. *Every weakly* $\overline{\theta}$*-refinable space has property* $B(D,(\omega_0)^2)$.

Proof. Let $\cup_{i=1}^{\infty}\mathcal{G}_i$ be a weak $\overline{\theta}$-cover of X where $\mathcal{G}_i = \{G(\alpha,i): \alpha \in A\}$. Let $G_k^* = \cup\mathcal{G}_k$ for each k and $\mathcal{G}^* = \{G_k^*\}_{k=1}^{\infty}$. Define for each $i \geq 1$ and $j \geq 1$,

$$P(i,j) = \{x \in X: \text{ord}(x,\mathcal{G}^*) < i \text{ or } \text{ord}(x,\mathcal{G}^*) = i \text{ and } 0 < \text{ord}(x,\mathcal{G}_k) \leq j \text{ for some } k\}$$

We show that for each (i,j) there exists a sequence of collections $\{\mathcal{F}_k\}_{k=1}^{\infty}$ such that $\mathcal{F}_k$ is a discrete closed collection in $X - P(i,j)$. Since $X = \cup_{i=1}^{\infty}\cup_{j=1}^{\infty}P(i,j)$ and $P(i,j+1) = P(i,j) \cup [\cup_{k=1}^{\infty}[\cup\mathcal{F}_k]]$ the proof will be complete. Let i and j be fixed.

Define, $H_i = \{x \in X: \text{ord}(x,\mathcal{G}^*) \leq i\}$.

$\mathcal{B}_k = \{B \subseteq A_k: |B| = j + 1\}$.

$S_k = \{x \in X: 0 < \text{ord}(x,\mathcal{G}_k) \leq j + 1\}$.

Now for each k and each $B \in \mathcal{B}_k$ let $F(B,k) = [\bigcap_{\alpha\in B} G(\alpha,k)] \cap [G_k^* \cap H_i \cap S_k]$ and $\mathcal{F}_k = \{F(B,k): B \in \mathcal{B}_k\}$.

We assert that $\mathcal{F}_k$ is a discrete closed collection in $X - P(i,j)$. Let k be fixed and $x \in X - P(i,j)$. Then $\text{ord}(x,\mathcal{G}^*) \geq i$.

(1) If $\text{ord}(x,\mathcal{G}^*) > i$, then $X - H_i$ is a neighborhood of x which intersects no member of $\mathcal{F}_k$.

(2) Suppose $\text{ord}(x,\mathcal{G}^*) = i$.

Case I. If $x \notin G_k^*$, then x belongs to exactly i other members $\{G_{\alpha_\ell}^* : \ell = 1,2,\cdots i\}$ of $\mathcal{G}^*$. Hence $\cap_{\ell=1}^{i}G_{\alpha_\ell}^*$ is a neighborhood of x which misses $G_k^* \cap H_i$ and hence intersects no member of $\mathcal{F}_k$.

Case II. Suppose $x \in G_k^*$. If $\text{ord}(x,\mathcal{G}_k) > j + 1$ then x belongs to at least j + 2 members of $\mathcal{G}_k$, say $G(\alpha_\ell,k)$ for $\ell = 1,\cdots j+2$. But $\cap_{\ell=1}^{j+2} G(\alpha_\ell,k) \cap S_k = \phi$, so $\cap_{\ell=1}^{j+2} G(\alpha_\ell,k)$ intersects no member of $\mathcal{F}_k$.

Finally if $\text{ord}(x,\mathcal{G}_k) = j + 1$ then x belongs to exactly j + 1 members of $\mathcal{G}_k$, $G(\alpha_\ell,k)$ for $\ell = 1,2,\cdots j+1$. Then $\cap_{\ell=1}^{j+1} G(\alpha_\ell,k)$ intersects only $F(B,k)$ where $B = \{\alpha_1,\alpha_2,\cdots\alpha_{j+1}\}$.

It is easy to see that $P(i,j+1) = P(i,j) \cup [\cup_{k=1}^{\infty}[\cup\mathcal{F}_k]]$ so that the proof is complete. Hence X has property $B(D,(\omega_0)^2)$.

Remark. It is important to note that in the construction above, the families $\mathcal{F}_k$ cover all points which have finite positive order with respect to some $\mathcal{G}_k$.

Lemma. If $\mathcal{U}$ be an open cover of a space X *and* C *a closed subset of* X. *Suppose that* $\mathcal{F} = \{F_\alpha : \alpha \in A\}$ *is a partial refinement of* $\mathcal{U}$ *such that*

(1) each member of $\mathcal{F}$ *is closed in* X - C *and*

(2) $\mathcal{F}$ *is locally finite on* X - C.

Then there exists a sequence of open collections $\{\mathcal{G}_i\}_{i=1}^{\infty}$ *which partially refined* $\mathcal{U}$, *such that each* $x \in [\cup\mathcal{F}] - C$ *has finite positive order with respect to some* $\mathcal{G}_k$. (In fact, $\text{ord}(x,\mathcal{G}_k) = 1$ for some k.)

Proof. Now if $\Gamma_n = \{B: B \subseteq A, |B| = n\}$, define $H(B) = \bigcap_{\beta \in B} F_\beta$, for each $B \in \Gamma_n$. Note that $H(B) \subseteq U(B)$ for some $U(B) \in \mathcal{U}$. Let $\mathcal{G}_n = \{G(B): B \in \Gamma_n\}$, where

$G(B) = [U(B) - C] - \cup\{H(B'): B' \in \Gamma$ and $B' \neq B\}$. Clearly $\mathcal{G}_n$ is a collection of open sets for each n. Furthermore if $x \in [\cup\mathcal{F}] - C$, then $\text{ord}(x,\mathcal{F}) = k$ for some k; so x belongs to ecactly $F_{\alpha_1}, F_{\alpha_2}, \cdots, F_{\alpha_k}$. Therefore $x \in G(B)$ only when $B = \{\alpha_1, \cdots, \alpha_k\}$. Hence $\text{ord}(x, \mathcal{G}_k) = 1$.

Theorem 2.3. *If a space* X *has property* $B(LF, (\omega_0)^2)$, *then* X *is weakly* θ-*refinable.*

Proof. Suppose X has property $B(LF, (\omega_0)^2)$ and $\mathcal{U}$ is an open cover of X. Then there exists a collection of families $\{\mathcal{F}_s: s < (\omega_0)^2\}$ such that

(i) each member of $\mathcal{F}_s$ is closed in $X - \bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$,

(ii) $\bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$ is closed in X for each s,

(iii) $\mathcal{F}_s$ is locally finite in $X - \bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$.

By the previous lemma, there exists for each s, a sequence $\{\mathcal{G}_i^s\}_{i=1}^{\infty}$ of open collections such that each point $x \in [\cup\mathcal{F}_s] - \bigcup_{s'<s} [\cup\mathcal{F}_{s'}]$ has finite positive order with respect to $\mathcal{G}_k^s$, for some k. Without loss of generality we may assume that each $\mathcal{G}_k^s$ is a partial refinement of $\mathcal{U}$. It is easy to see that $\{\bigcup_{i<\omega_0} \bigcup_{s<(w_0)^2} \mathcal{G}_i^s\}$ is a weak θ-refinement of $\mathcal{U}$, and hence X is weakly θ-refinable.

Remark. It should be noted that Theorem 2.3 above remains true for any countable ordinal β. The proof is similar.

Summary. Property $B(D,\omega_0) \Rightarrow$ weakly $\overline{\theta}$-refinable $\Rightarrow$ property $B(D,\omega_0)^2) \Rightarrow$ property $B(LF,(\omega_0)^2) \Rightarrow$ weakly θ-refinable.

3. Property B (HCP, α) and Irreducibility

In [17] the author obtained the following result.

Theorem 3.1. *Every weak $\overline{\theta}$-refinable space is irreducible.*

Since property $B(D,\omega_0) \Rightarrow$ weakly $\overline{\theta}$-refinable, every space with property $B(D,\omega_0)$ is irreducible. Here we can obtain the stronger result, that every space with property $B(HCP,\alpha)$ is irreducible.

The following lemmas are straightforward, and hence their proofs are omitted.

Lemma 3.2. *Let* $H \subseteq X$ *and let* $\mathcal{U}$ *be a collection of open sets in* X *which covers* H. *If* $\mathcal{U}|H$ *has a minimal open* (in H) *refinement then there exists an open* (in X) *collection* $\mathcal{V}$ *which partially refines* $\mathcal{U}$ *and covers* H, *such that* $\mathcal{V}$ *is a minimal open cover of* $\cup\mathcal{V}$.

Lemma 3.3. *Let* X *be a topological space and* $H = \bigcup_{s<\alpha} H_s$ *where* $\bigcup_{s'<s} H_{s'}$ *is a closed subset of* X *for each* $s < \alpha$. *Let* $\mathcal{U}$ *be a collection of open subsets of* X *which covers* H. *If for each* $s < \alpha$, $\mathcal{W}_s$ *is a collection of open subsets of* X *which partially refines* $\mathcal{U}$ *and covers* $H_s - \bigcup_{s'<s} [\cup\mathcal{W}_{s'}]$ *minimally, then there exists a collection* $\mathcal{V}$ *of open subsets of* X *which partially refines* $\mathcal{U}$, *covers* H, *and is a minimal open cover of* $\cup\mathcal{V}$.

Theorem 3.4. *Let* $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ *be a collection of open subsets of a space* X *and* $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ *a hereditarily* *closure preserving collection such that* $H_\alpha \subseteq U_\alpha$ *for each* $\alpha \in A$. *Then* $\mathcal{U}$ *has an open partial refinement which covers* $\cup\mathcal{H}$ *and is a minimal open cover of its union.*

Proof. Suppose that $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ is a hereditarily closure preserving collection with $H_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. We assume that A is well ordered. For each $\alpha \in A$ choose

$$x_\alpha \in H_\alpha - \bigcup_{\beta<\alpha} H_\beta \text{ when } H_\alpha - \bigcup_{\beta<\alpha} H_\beta \neq \phi,$$

and let $A' = \{\alpha \in A: H_\alpha - \bigcup_{\beta<\alpha} H_\beta \neq \phi\}$. Since X is T_1 and $\mathcal{H}$ is hereditarily closure preserving $\{x_\alpha: \alpha \in A'\}$ is a discrete closed collection in X. Define

$$W_\alpha = U_\alpha - \cup\{x_\beta: \beta \in A' \text{ and } \beta \neq \alpha\} \text{ for each } \alpha \in A.$$

Clearly $\mathcal{W} = \{W_\alpha: \alpha \in A'\}$ is a minimal open cover of $\cup\mathcal{H}$.

We now can obtain the following.

Theorem 3.5. *Every space* X *space with property* B(HCP,α) *is irreducible, for any ordinal* α.

Proof. Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a refinement $\bigcup_{s<\alpha} \mathcal{F}_s$ satisfying properties in Definition 1.6 above. By induction we construct a sequence of $\{\mathcal{V}_s\}_{s<\alpha}$ of open collections such that for each $s < \alpha$,

(i) $\mathcal{V}_s$ is a partial refinement of $\mathcal{U}$,

(ii) $\bigcup_{s'\leq s} \mathcal{V}_{s'}$ covers $\bigcup_{s'\leq s} [\cup\mathcal{F}_{s'}]$

(iii) $\bigcup_{s'\leq s} \mathcal{V}_{s'}$ is a minimal open cover of its union.

(1) For $s = 1$, $\mathcal{F}_1$ is a hereditarily closure preserving collection of closed subsets of X. By Theorem 3.4 above there exists an open partial refinement $\mathcal{V}_1$ of $\mathcal{U}$ such that $\mathcal{V}_1$ is a minimal open cover of $\cup\mathcal{F}_1$.

(2) Assume that $\mathcal{V}_{s'}$ has been constructed satisfying (i)-(iii) above for s' < s. Define $\mathcal{F}_s^* = \{F - \bigcup_{s'<s} [\cup\mathcal{V}_{s'}]: F \in \mathcal{F}_s\}$ so that $\mathcal{F}_s^*$ is a hereditarily closure preserving collection in X. By Theorem 3.4 again there exists an open partial refinement $\mathcal{W}_s$ of $\mathcal{U}$ such that $\mathcal{W}_s$ covers $\cup\mathcal{F}_s^*$ and is a minimal open cover of its union. Now define $\mathcal{V}_s = \{W - \bigcup_{s'<s} [\cup\mathcal{F}_{s'}]: W \in \mathcal{W}_s\}$. It is easy to check that $\mathcal{V}_s$ satisfies properties (i)-(iii) above and the induction is complete. As in Lemma 3.3 $\bigcup_{s<\alpha} \mathcal{V}_s$ is a minimal open cover of X and refines $\mathcal{U}$. Hence X is irreducible.

Corollary 3.6. *Every* $\aleph_1$*-compact space with property* B(HCP,α) *is Lindelöf, where* α *is any countable ordinal.*

Theorem 3.7. *Let* f: X → Y *be a closed continuous map. If* X *has property* B(HCP,α), *then* Y *has property* B(HCP,α) *and hence is irreducible.*

Proof. The proof follows from the fact that closure preserving collections are preserved under closed maps.

4. Applications and Shrinkability

Definition 4.1. An open cover $\{G_\alpha: \alpha \in A\}$ is *shrinkable* if there exists a closed cover $\{F_\alpha: \alpha \in A\}$ such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.

In [19] the author obtained the following result.

Theorem 4.2. *A space* X *is normal iff every weak* $\overline{\theta}$*-cover of* X *is shrinkable.*

A generalization of this result can now be proved using the notion of property above.

Theorem 4.3. *Let* $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ *be an open cover of a space* X. *If* k *is any countable ordinal, and* $\mathcal{G}$ *has an open refinement* $\bigcup_{s<k} \mathcal{V}_s$ *where* $\mathcal{V}_s = \{V(\alpha,s): \alpha \in A\}$ *satisfies,*

(1) $\overline{V(\alpha,s)} \subseteq G_\alpha$ *for each* $\alpha \in A$,

(2) $\bigcup_{\alpha \in A} V(\alpha,s)$ *is a cozero set in* X *for each* s,

then $\mathcal{G}$ *is shrinkable.*

Proof. Define $V_s^* = \bigcup_{\alpha \in A} V(\alpha,s)$ for each $s < k$ so that $\{V_s^*: s < k\}$ is a countable cozero cover of X. Then $\{V_s^*: s < k\}$ has a locally finite open refinement $\{W_s^*: s < k\}$ such that $W_s^* \subseteq V_s^*$ for each $s < k$. Define $H(\alpha,s) = W_s^* \cap V(\alpha,s)$ for each $\alpha \in A$ and each $s < k$, and $H_\alpha = \bigcup_{s<k} H(\alpha,s)$. It should be clear that $\overline{H}_\alpha \subseteq G_\alpha$ for each $\alpha \in A$ and $\{H_\alpha: \alpha \in A\}$ covers X. Hence $\mathcal{G}$ is shrinkable.

Theorem 4.4. *Let* X *be a normal space. For any countable ordinal k, every open cover with property* B(HCP,k) *is shrinkable.*

Proof. Let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be an open cover of X with property B(HCP,k) where k is any countable ordinal. Then $\mathcal{G}$ has a refinement $\bigcup_{s<k} \mathcal{F}_s$ where,

(1) $\mathcal{F}_s = \{F(\alpha,s): \alpha \in A\}$ is HCP and closed in $X - \bigcup_{s'<s} [\bigcup \mathcal{F}_{s'}]$.

(2) $F(\alpha,s) \subseteq G_\alpha$ for each $\alpha \in A$.

We show by transfinite induction that there exists for each $s < k$, an open collection $\mathcal{V}_s = \{V(\alpha,s): \alpha \in A\}$ satisfying

(1) $V(\alpha,s) \subseteq \overline{V(\alpha,s)} \subseteq G_\alpha$ for each $\alpha \in A$,

(2) $\bigcup_{\alpha \in A} V(\alpha,s)$ is cozero in X for each s.

(3) $\bigcup_{s' \leq s} \mathcal{V}_s$ covers $\bigcup_{s' \leq s} \mathcal{F}_s$ for each s.

Assume $\mathcal{V}_{s'}$ with the above properties has been constructed for all $s' < s$. Define $H(\alpha,s) = F(\alpha,s) - \bigcup[\bigcup_{s'<s} \mathcal{V}_s]$ so that $H(\alpha,s) = \overline{H(\alpha,s)} \subseteq G_\alpha$ for each $\alpha \in A$. Since $\mathcal{H} = \{H(\alpha,s): \alpha \in A\}$ is closure preserving and X is normal, there exists an open collection $\mathcal{V}_s = \{V(\alpha,s): \alpha \in A\}$ such that $\mathcal{V}_s$ is a partial refinement of $\mathcal{G}$, and

(1) $H(\alpha,s) \subseteq V(\alpha,s) \subseteq \overline{V(\alpha,s)} \subseteq G_\alpha$ for each $\alpha \in A$,

(2) $\bigcup_{\alpha \in A} V(\alpha,s)$ is a cozero set in X.

Clearly $\bigcup_{s' \leq s} \mathcal{V}_{s'}$ covers $\bigcup_{s' \leq s} \mathcal{F}_s$ and the construction is complete. By Theorem 4.3 above, $\mathcal{G}$ is shrinkable.

Theorem 4.5. *Suppose that* $X = \bigcup_{i=1}^\infty H_i$ *where each* $H_i = \overline{H}_i$ *has property* $B(D,\omega_0)$. *Then* X *has property* $B(D,\omega_0)$.

Proof. Suppose each H_i has property $B(D,\omega_0)$ and $\mathcal{U}$ is an open cover of X. Then $\mathcal{U}/H_i$ has a refinement $\bigcup_{j=1}^\infty \mathcal{F}_j^i$ such that $\mathcal{F}_j^i$ is a discrete closed collection in $H_i = \bigcup_{k<j} \mathcal{F}_k^i$. Since $\mathcal{F}_1^i$ is a discrete closed collection in X for each i, the natural diagonalization of the families $\bigcup_{i=1}^\infty \bigcup_{j=1}^\infty \mathcal{F}_j^i$ yields the desired collections satisfying property $F(D,\omega_0)$.

Theorem 4.6. *Let* $f: X \to Y$ *be a perfect map.*

(1) If X *has property* $B(LF,\alpha)$*, then so does* Y *and hence* Y *is irreducible.*

(2) If X *is weakly* $\overline{\theta}$*-refinable, then* Y *has property* $B(LF,(\omega_0)^2)$ *and hence is weak* θ*-refinable.*

Open Questions.

(1) Is weak $\overline{\theta}$-refinability or weak θ-refinability preserved under perfect or closed maps?

(2) Is metacompactness equivalent to weak θ-refinable, almost expandable and orthocompactness?

(3) When are weakly θ-refinable spaces irreducible? For example, is countably metacompactness enough?

(4) When does property $B(D,(\omega_0)^2)$ imply weak $\overline{\theta}$-refinability?

(5) Is there a simple example of a space which has property $B(D,\omega_0+1)$ but does not have property $B(D,\omega_0)$?

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