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THE NUMBER OF COMPACT FIBERS OF MAPPINGS ON BANACH SPACES

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This note effectively answers the following question: If B is an infinite dimensional Banach space and $f: B \to \mathbf{R}^n$ is a continuous mapping, how many compact fibers can f have? We show that for n finite there are at most 2^n compact fibers and that there exists a continuous function with exactly 2^n compact fibers.

1. Upper Bounds for the Number of Compact Fibers

We recall that if $f: A \to B$ is a function, a fiber of f is a set of the form $f^{-1}(b)$, where $b \in B$.

Theorem 1.1. *Suppose* $(H, \langle -,- \rangle)$ *is an inner-product space with infinite orthogonal dimension* α, *and* $f: H \to \mathbf{R}$ *is a continuous function. If* $[a,c] \subseteq f[H]$, *then for each* $b \in (a,c)$, *the fiber* $f^{-1}(b)$ *contains a closed discrete subset of cardinal* α.

Proof. Assume $a < b < c$ and choose $p \in f^{-1}(a)$ and $q \in f^{-1}(c)$. Choose $\delta > 0$ such that $\|x-p\| < \delta$ implies $f(x) < b$ and $\|x-q\| < \delta$ implies $f(x) > b$. Let $r = p - q$ and choose an orthonormal basis $E = \{\bar{e}_i : i \in I\}$ for the orthogonal complement in H of the linear subspace generated by r. For each i, let $e_i = \delta\bar{e}_i$, $s_i = p + e_i$, and $t_i = q + e_i$. Since $f(s_i) < b$ and $f(t_i) > b$, it follows from the intermediate value theorem that for each $i \in I$, there exists a point p_i on the line segment joining s_i and t_i such that $f(p_i) = b$. We claim that the set $\{p_i: i \in I\}$ is uniformly discrete. To see this, write $p_i = \gamma_i s_i + (1-\gamma_i)t_i$ with $0 < \gamma_i < 1$. Then for $i \neq j$, $\|p_i-p_j\|^2 = \langle p_i-p_j, p_i-p_j\rangle = (\gamma_i-\gamma_j)^2 r^2 + \langle e_i-e_j, e_i-e_j\rangle = (\gamma_i-\gamma_j)^2 r^2 + 2\delta^2$.

Since a recent theorem of Toruňcuk asserts that every Banach space is homeomorphic to a Hilbert space, we obtain the following corollary of 1.1.

Corollary 1.2. *Suppose* B *is an infinite dimensional Banach space with density character* α*, and suppose* $f: B \to R$ *is a continuous mapping. Then* f *has at most two fibers of density character less than* α*, and* f *has at most two compact fibers.*

Remarks.

(i) The preceding argument actually shows that if a continuous real-valued function $f: B \to \mathbf{R}$ has two compact fibers $f^{-1}(a)$ and $f^{-1}(c)$ with $a < c$, then f is bounded with $a = \inf f[B]$, $c = \sup f[B]$; hence an unbounded continuous real-valued function has at most one compact fiber and an onto continuous real-valued function has no compact fibers.

(ii) Professor William Transue of Auburn University has informed the authors that he can establish 1.2 for infinite-dimensional normed linear spaces. Professor Transue has also established that for metric spaces the condition "every continuous real-valued function has at most two compact fibers" is equivalent to an order condition on fibers of continuous functions with compact zero sets.

The following Proposition is an immediate corollary of 1.2.

Proposition 1.3. *If* B *is an infinite-dimensional Banach space and* α *is a cardinal number, then a continuous function* $f: B \to \mathbf{R}^{\alpha}$ *has at most* 2^{α} *compact fibers.*

Proposition 1.4. *The following conditions are equivalent for a metric space* X.

(i) Every continuous function $f: X \to \mathbf{R}$ *has at most two compact fibers.*

(ii) If $f: X \to \mathbf{R}$ *is continuous and* $f^{-1}(0)$ *is compact, then either* $f \geq 0$ *or* $f \leq 0$.

Proof. (i) $\Rightarrow$ (ii). Assume that there exists a continuous function $f: X \to \mathbf{R}$ such that $K = f^{-1}(0)$ is compact and $f(p) < 0 < f(q)$ for points p and q in X. Without loss of generality we may assume $f[X] \subseteq [f(p), f(q)]$--otherwise replace f by $(f \vee f(p)) \wedge f(q)$. Choose a continuous function $g: X \to [0,1]$ such that $g^{-1}(0) = K$ and $g^{-1}(1) = \{p,q\}$. Let $h = fg$ be the pointwise product of h and g. Then $h^{-1}(0) = K$, $h(p) = f(p)$, $h(q) = f(q)$, and if $x \notin K \cup \{p,q\}$, then $f(p) \leq f(x) \leq f(q)$ and $0 < g(x) < 1$, so $f(p) < h(x) < f(q)$. Therefore, h has three compact fibers: $K = h^{-1}(0)$, $\{p\}=h^{-1}(f(p))$, and $\{q\}= h^{-1}(f(q))$. (ii) $\Rightarrow$ (i). If the continuous function $f: X \to R$ has three compact fibers, say $f^{-1}(a)$, $f^{-1}(b)$, and $f^{-1}(c)$, where $a < b < c$, the function $f-b$ violates condition (ii).

2. Lower Bounds for the Number of Compact Fibers

In this section we consider conditions on a space X which assure that for a pair of cardinal numbers (α,β), there exists a continuous function $f: X \to \mathbf{R}^{\alpha}$ with at least β compact fibers.

Recall that a topological space is perfectly normal if every closed subset is a fiber of a continuous real-valued function. In the following theorem we view α as an initial ordinal and an ordinal as the set of its pro-decessors.

Theorem 2.1. *Suppose that the perfectly normal space* X *has a closed discrete subset of cardinality* 2^{α}. *Then there exists a continuous function* $f: X \to \mathbf{R}^{\alpha}$ *such that* f *has at least* 2^{α} *compact fibers.*

Proof. Let D be a closed discrete subset of X such that $\text{card}(D) = 2^{\alpha}$. For each function $\sigma: \alpha \to \{0,1\}$, choose $p_{\sigma} \in D$ such that $p_{\sigma} \neq p_{\sigma'}$ for $\sigma \neq \sigma'$. For each $s \in \alpha$ define the sets $A_s = \{p_{\sigma}: \sigma(s) = 0\}$ and $B_s = \{p_{\sigma}: \sigma(s) = 1\}$. It follows from the perfect normality of X that there exists a continuous function $f_s: X \to \mathbf{R}$ such that $f_s^{-1}(0) = A_s$ and $f_s^{-1}(1) = B_s$. Define $f: X \to \mathbf{R}^{\alpha}$ by $f(x) = (f_s(x))_{s\in\alpha}$. Then for each σ, the fiber $f^{-1}(\sigma)$ is the singleton set $\{p_{\sigma}\}$, so f has 2^{α} compact fibers.

Corollary 2.2. *Let* X *be an infinite metric space. Then for each* $n = 1,2,\cdots$, *there exists a continuous function* $f: X \to \mathbf{R}^{n}$ *with at least* 2^{n} *compact fibers.*

Combining 1.3 and 2.2 we get the following.

Corollary 2.3. *If* X *is an infinite dimensional Banach space and* n *is a positive integer, then there exists a continuous function* $f\colon X \to R^n$ *with exactly* 2^n *compact fibers.*

Remark. Using the fact that any non-compact metric space has an infinite closed discrete subset and the argument 2.1, one can show that every infinite metric space X admits a continuous function $f\colon X \to R^{\omega_0}$ with infinitely many compact fibers.

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