Research Announcement:

THE NUMBER OF COMPACT FIBERS OF MAPPINGS ON BANACH SPACES

by

Ronnie Levy and M. D. Rice

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This note effectively answers the following question:
If B is an infinite dimensional Banach space and $f: B \to \mathbf{R}^n$
is a continuous mapping, how many compact fibers can f
have? We show that for n finite there are at most 2^n
compact fibers and that there exists a continuous function
with exactly 2^n compact fibers.

1. Upper Bounds for the Number of Compact Fibers

We recall that if $f: A \to B$ is a function, a fiber of
f is a set of the form $f^{-1}(b)$, where $b \in B$.

*Theorem 1.1. Suppose $(H, \langle -, - \rangle)$ is an inner-product
space with infinite orthogonal dimension α, and $f: H \to \mathbf{R}$
is a continuous function. If $[a,c] \subseteq f[H]$, then for each
$b \in (a,c)$, the fiber $f^{-1}(b)$ contains a closed discrete
subset of cardinal α.*

Proof. Assume $a < b < c$ and choose $p \in f^{-1}(a)$ and
$q \in f^{-1}(c)$. Choose $\delta > 0$ such that $\| x-p \| < \delta$ implies
$f(x) < b$ and $\| x-q \| < \delta$ implies $f(x) > b$. Let $r = p - q$
and choose an orthonormal basis $E = \{\bar{e}_i : i \in I\}$ for the
orthogonal complement in H of the linear subspace generated
by r. For each i, let $e_i = \delta \bar{e}_i$, $s_i = p + e_i$, and $t_i = q + e_i$.
Since $f(s_i) < b$ and $f(t_i) > b$, it follows from the inter-
mediate value theorem that for each $i \in I$, there exists a

point p_i on the line segment joining s_i and t_i such that
$f(p_i) = b$. We claim that the set $\{p_i : i \in I\}$ is uniformly
discrete. To see this, write $p_i = \gamma_i s_i + (1-\gamma_i) t_i$ with
$0 < \gamma_i < 1$. Then for $i \neq j$, $\| p_i - p_j \|^2 = \langle p_i - p_j, p_i - p_j \rangle =$
$(\gamma_i - \gamma_j)^2 r^2 + \langle e_i - e_j, e_i - e_j \rangle = (\gamma_i - \gamma_j)^2 r^2 + 2\delta^2$.

Since a recent theorem of Torunčuk asserts that every
Banach space is homeomorphic to a Hilbert space, we obtain
the following corollary of 1.1.

*Corollary 1.2. Suppose B is an infinite dimensional
Banach space with density character α, and suppose
f: B $\to$ R is a continuous mapping. Then f has at most two
fibers of density character less than α, and f has at
most two compact fibers.*

Remarks.

(i) The preceding argument actually shows that if a
continuous real-valued function f: B $\to$ R   has two compact
fibers $f^{-1}(a)$ and $f^{-1}(c)$ with $a < c$, then f is bounded with
$a = \inf f[B]$, $c = \sup f[B]$; hence an unbounded continuous
real-valued function has at most one compact fiber and an
onto continuous real-valued function has no compact fibers.

(ii) Professor William Transue of Auburn University
has informed the authors that he can establish 1.2 for
infinite-dimensional normed linear spaces. Professor
Transue has also established that for metric spaces the
condition "every continuous real-valued function has at
most two compact fibers" is equivalent to an order condition
on fibers of continuous functions with compact zero sets.

The following Proposition is an immediate corollary
of 1.2.

*Proposition 1.3.   If B is an infinite-dimensional
Banach space and α is a cardinal number, then a continuous
function f: B → $\mathbf{R}^\alpha$ has at most 2^α compact fibers.*

*Proposition 1.4.   The following conditions are equiva-
lent for a metric space X.*

*(i) Every continuous function f: X → $\mathbf{R}$ has at most
two compact fibers.*

*(ii) If f: X → $\mathbf{R}$ is continuous and $f^{-1}(0)$ is compact,
then either f ≥ 0 or f ≤ 0.*

Proof. (i) ⇒ (ii). Assume that there exists a
continuous function f: X → $\mathbf{R}$ such that $K = f^{-1}(0)$ is compact
and $f(p) < 0 < f(q)$ for points p and q in X. Without loss
of generality we may assume $f[X] \subseteq [f(p), f(q)]$--otherwise
replace f by $(f \vee f(p)) \wedge f(q)$. Choose a continuous func-
tion g: X → [0,1] such that $g^{-1}(0) = K$ and $g^{-1}(1) = \{p, q\}$.
Let h = fg be the pointwise product of h and g. Then
$h^{-1}(0) = K$, $h(p) = f(p)$, $h(q) = f(q)$, and if $x \notin K \cup \{p, q\}$,
then $f(p) \le f(x) \le f(q)$ and $0 < g(x) < 1$, so $f(p) < h(x) <$
$f(q)$. Therefore, h has three compact fibers: $K = h^{-1}(0)$,
$\{p\} = h^{-1}(f(p))$, and $\{q\} = h^{-1}(f(q))$. (ii) ⇒ (i). If the
continuous function f: X → R has three compact fibers, say
$f^{-1}(a)$, $f^{-1}(b)$, and $f^{-1}(c)$, where a < b < c, the function
f-b violates condition (ii).

2. Lower Bounds for the Number of Compact Fibers

In this section we consider conditions on a space X which assure that for a pair of cardinal numbers (α, β), there exists a continuous function $f: X \to \mathbb{R}^\alpha$ with at least β compact fibers.

Recall that a topological space is perfectly normal if every closed subset is a fiber of a continuous real-valued function. In the following theorem we view α as an initial ordinal and an ordinal as the set of its predecessors.

Theorem 2.1. Suppose that the perfectly normal space X has a closed discrete subset of cardinality 2^α. Then there exists a continuous function $f: X \to \mathbb{R}^\alpha$ such that f has at least 2^α compact fibers.

Proof. Let D be a closed discrete subset of X such that $card(D) = 2^\alpha$. For each function $\sigma: \alpha \to \{0,1\}$, choose $p_\sigma \in D$ such that $p_\sigma \neq p_{\sigma'}$ for $\sigma \neq \sigma'$. For each $s \in \alpha$ define the sets $A_s = \{p_\sigma : \sigma(s) = 0\}$ and $B_s = \{p_\sigma : \sigma(s) = 1\}$. It follows from the perfect normality of X that there exists a continuous function $f_s: X \to \mathbb{R}$ such that $f_s^{-1}(0) = A_s$ and $f_s^{-1}(1) = B_s$. Define $f: X \to \mathbb{R}^\alpha$ by $f(x) = (f_s(x))_{s \in \alpha}$. Then for each σ, the fiber $f^{-1}(\sigma)$ is the singleton set $\{p_\sigma\}$, so f has 2^α compact fibers.

Corollary 2.2. Let X be an infinite metric space. Then for each $n = 1, 2, \cdots$, there exists a continuous function $f: X \to \mathbb{R}^n$ with at least 2^n compact fibers.

Combining 1.3 and 2.2 we get the following.

Corollary 2.3. If X *is an infinite dimensional Banach space and* n *is a positive integer, then there exists a continuous function* f: $X \to R^n$ *with exactly* 2^n *compact fibers.*

Remark. Using the fact that any non-compact metric space has an infinite closed discrete subset and the argument 2.1, one can show that every infinite metric space X admits a continuous function f: $X \to R^{\omega_0}$ with infinitely many compact fibers.

George Mason University

Fairfax, Virginia 22030