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Research Announcement:

U-EMBEDDED SUBSETS OF THE PLANE

by

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For a metric space M, let U(M) denote the set of all real-valued uniformly continuous functions on M and let U*(M) denote the set of bounded members of U(M). If X is a subset of M, then X is U-embedded [respectively U*-embedded] in M if every element of U(X) [respectively U*(X)] is the restriction to X of an element of U(M). The following theorem of Katětov answers all questions about U*-embedding.

Theorem. (Katětov) *If* M *is a metric space, every subset of* M *is* U**-embedded in* M.

The question of which subsets of a metric space are U-embedded is much more difficult. Not every subset of an arbitrary metric space is U-embedded--the set N of natural numbers is not U-embedded in the space $\mathbf{R}$ of real numbers because the uniformly continuous function $f: N \to \mathbf{R}$ given by $f(n) = n^2$ does not extend to a uniformly continuous function on $\mathbf{R}$. This example provides the key to characterizing the U-embedded subsets of $\mathbf{R}$.

Theorem 1. X *is* U-*embedded in* $\mathbf{R}$ *if and only if* X *is not the union of an infinite uniformly discrete family of non-empty subsets.*

Unlike the case of $\mathbf{R}$, the U-embedded subsets of $\mathbf{R}^2$ have not been characterized. The next theorem gives a class of subsets which are not U-embedded in $\mathbf{R}^2$, as well as a class of subsets which are U-embedded in $\mathbf{R}^2$.

Theorem 2. *Suppose* M *is a normed linear space and* $X \subseteq M$.

a) If X *is the union of an infinite uniformly discrete family of subsets, then* X *is not* U-*embedded in* M.

b) If X *is convex, then* X *is* U-*embedded in* M.

Since the convex subsets of $\mathbf{R}^2$ are U-embedded, it is natural to ask which starlike regions of $\mathbf{R}^2$ are U-embedded. The answer to even this question is not known. The following examples indicate the types of problems which can arise. Let $(a_n)_{n=1}^{\infty}$ be an unbounded increasing sequence of numbers and let $X = \cup_{k=0}^{\infty} L_k$, where L_0 is the non-negative x-axis and for $k = 1,2,\cdots$, L_k is the segment joining the origin to $(a_k,1)$. The sequences $a_n = n$ and $a_n = 2^n$ define the spaces X_1 and X_2 in Figure 1. Then X_1 is U-embedded in $\mathbf{R}^2$, but X_2 is not U-embedded in $\mathbf{R}^2$. The proof that X_1 is U-embedded combines the following facts: (i) L_0 is U-embedded in $\mathbf{R}^2$. (Theorem 2b.) (ii) X_1 is U*-embedded in R^2 (Katětov's theorem). (iii) If $f \in U(X_1)$ and the restriction of f to L_0 is bounded, then $f \in U^*(X_1)$. The fact that X_2 is not U-embedded is established by showing that the function $f\colon X_2 \to \mathbf{R}$ defined by

$$f(x,y) = \begin{cases} 0 \text{ if } y \le 1/2 \\ a_n \text{ if } (x,y) = (a_n,1) \\ \text{linear between } (a_n/2,1/2) \text{ and } (a_n,1) \end{cases}$$

is uniformly continuous. (See Figure 2.) By condition

(iii) above, f cannot be extended to an element of $U(X_1)$, so f certainly cannot be extended to an element of $U(R^2)$. By generalizing these arguments, one can establish the following.

Theorem 3. *If* X *is the star-like region defined above by the sequence* (a_n), *then* X *is* U-*embedded in* $\mathbf{R}^2$ *if and only if* $\lim_{n\to\infty} a_n/a_{n+1}$ *(exists and)* = 1.

References

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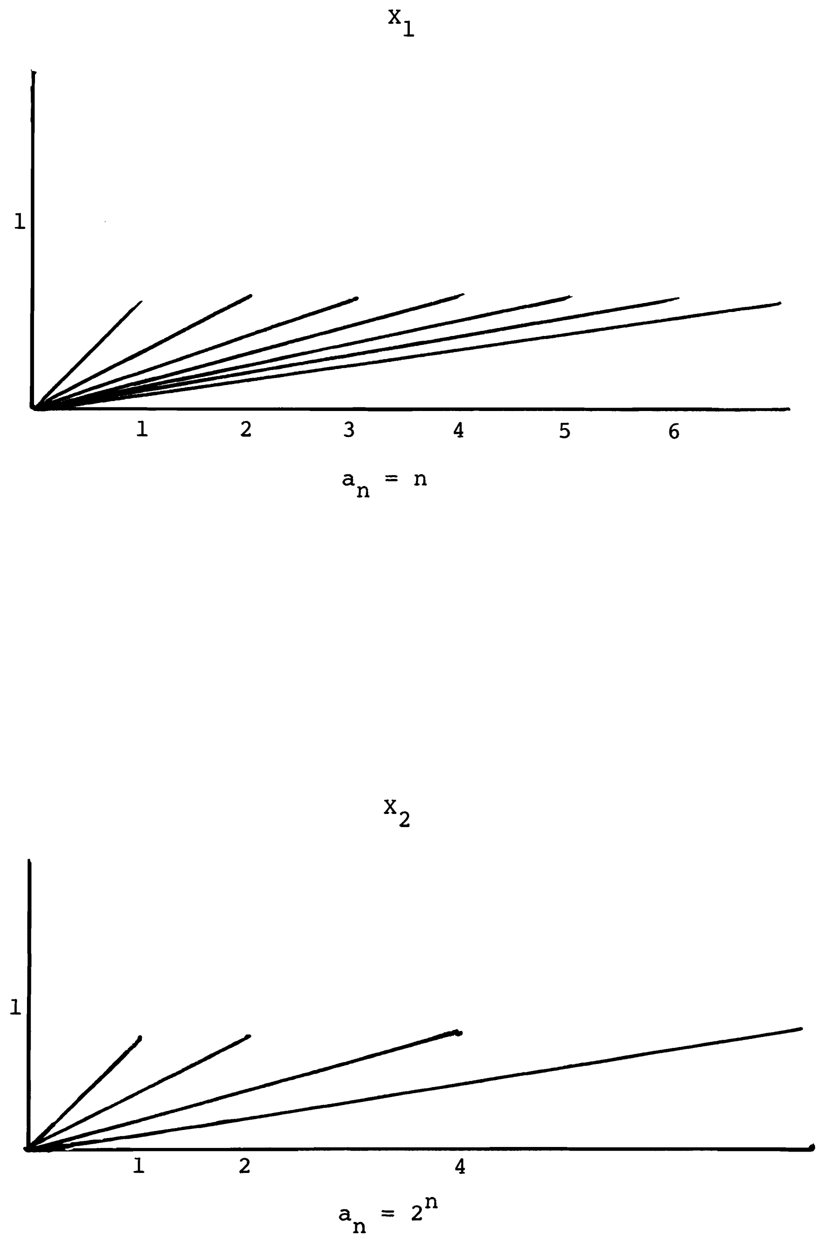
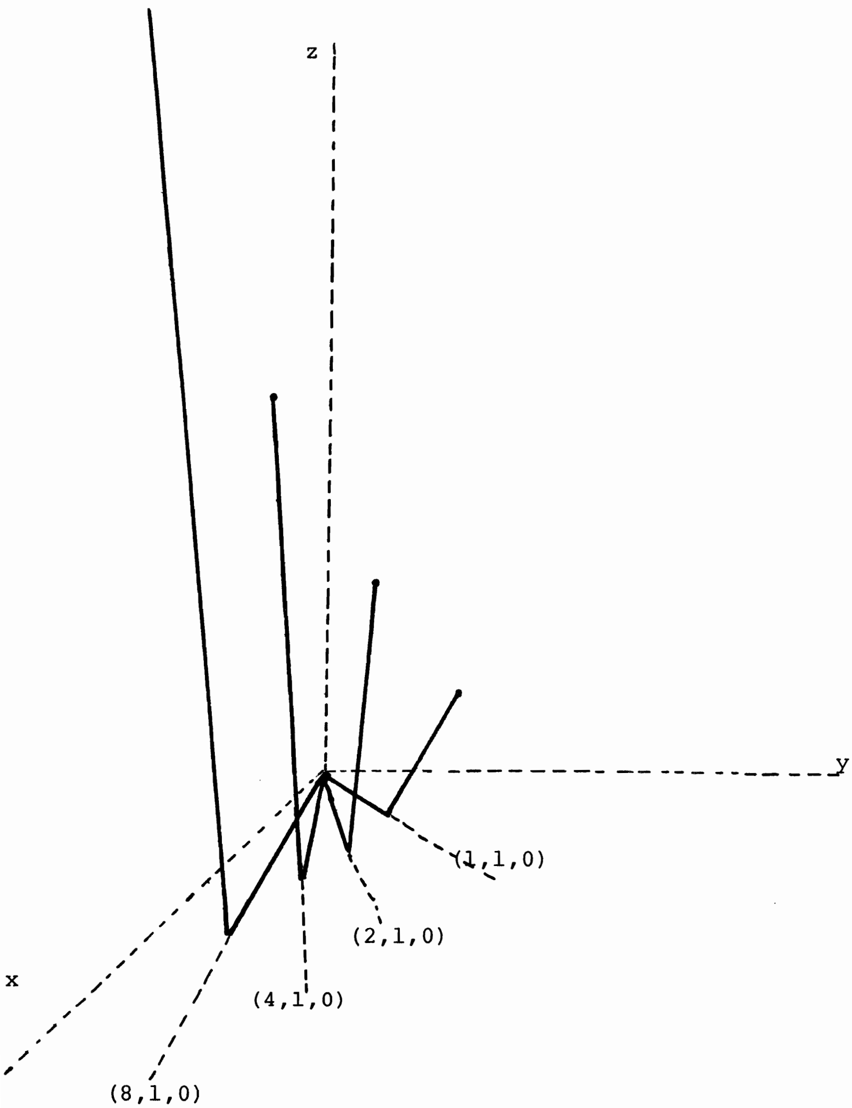


FIGURE 1



Graph of f: $X_2 \to \mathbf{R}$

FIGURE 2