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by

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TUNNELS, TIGHT GAPS, AND COUNTABLY COMPACT EXTENSIONS OF $\mathbb{N}$

Peter J. Nyikos

A fascinating unsolved problem of set-theoretic topology is whether there exists a separable, first countable, countably compact, noncompact (or non-normal) space. ["Space" will always mean "Hausdorff space"; but it is not hard to show that every countably compact, first countable, Hausdorff space is regular.] For the sake of convenience, we will call a countably compact space "nice" if it is both separable and first countable. The problem is richly intertwined with set theory. There are numerous examples under various set-theoretic hypotheses already, and in this announcement I will introduce several more. The following results have been around for some time.

Theorem 1. [2, in effect] $\neg P(\omega_2)$ *is equivalent to the statement that there exists a "nice" countably compact normal space whose set of nonisolated points is homeomorphic to* ω_1.

Theorem 2. (E. K. van Douwen) *If* BF(c), *then there exists a "nice" countably compact noncompact scattered space.*

Given a cardinal κ, $P(\kappa)$ is the statement that if $\mathcal{F}$ is a collection of subsets of ω which forms a subbase for a free filter, and $|\mathcal{F}| < \kappa$, then there is an infinite subset A of ω which is almost contained in every member of $\mathcal{J}$. [A set A is "almost contained" in a set B if $A \setminus B$ is finite.] The axiom BF(κ) substitutes functions from ω to ω for subsets of ω and "almost above," $f \leq^* g$, for "almost contained in." [We define $f \leq^* g$ to mean that $f(i) \leq g(i)$ for all but finitely many $i \in \omega$.] It is easy to show that P(κ) implies BF(κ); hence we have:

Corollary. If $c = \aleph_1$, *or* $c = \aleph_2$, *there exists a "nice" countably compact noncompact scattered space.*

Indeed, if $c = \aleph$ then we have BF(c); if $c = \aleph_2$, and $\neg P(\omega_2)$, then we apply Theorem 1, noting that such a space must be scattered; while if $P(\omega_2)$ then $BF(\omega_2)$, hence BF(c), etc.

A more difficult result is that the above corollary is also true if "noncompact" is replaced by "non-normal." In the $\neg P(\omega_2)$ part, the key result (see Theorem 9 below) is that the "Hausdorff gap" example of van Douwen in [1] can be made countably compact if (and only if) $\neg P(\omega_2)$. This example of van Douwen's is a space whose set of nonisolated points consists of two disjoint copies of ω_1 which cannot be put into disjoint open sets. In the BF(c) part, one begins with a version of this space and adds points as in van Douwen's argument for Theorem 2, until one obtains a countably compact space even if the starting space was not countably compact.

Theorem 3. *If* $\neg P(\omega_2)$ *or* BF(c), *then there exists a "nice" countably compact, non-normal scattered space.*

Corollary. If $c = \aleph_1$, *or* $c = \aleph_2$, *there exists a "nice" countably compact, non-normal scattered space.*

The reason for the emphasis on "scattered" is that it implies the space has a dense set of isolated points; in other words, it is a countably compact extension of $\mathbf{N}$. All spaces considered in the paper are of this sort, though not all will be scattered. Those most directly obtained from the following axioms are not scattered.

The Complete Tunnel Axiom. There is a continuous map from $\beta\mathbf{N} - \mathbf{N}$ *onto a* LOTS, *such that the preimage of every point has empty interior.*

Theorem 4. *The complete tunnel axiom is equivalent to the assertion that there is a compactification of* $\mathbf{N}$ *with ordered remainder, such that no sequence from* $\mathbf{N}$ *converges.*

The complete tunnel axiom obviously implies:

The ω_1*-Tunnel Axiom. There is a continuous map from* $\beta\mathbf{N} - \mathbf{N}$ *onto a non-first countable* LOTS, *which has the property that the preimage of every point without a countable base has empty interior.*

Theorem 5.* *The* ω_1*-tunnel axiom implies that there is a "nice" countably compact non-normal space.*

Theorem 6. CH $\Rightarrow$ P(c) + *Complete Tunnel Axiom* $\Rightarrow$ *there is a compactification of* Ψ *with ordered remainder and* 2^c *points.*

*See correction at end of article.

The problem of whether Ψ has a compactification of more than c points is still not completely solved. Theorem 6 may be the first consistency result on it.

The "tunnel" terms come from the following concepts.

Definition 1. Let X be a space and let κ be an infinite regular cardinal number. A *κ-tunnel through* X is a chain $\mathcal{C}$ of open subsets of X such that:

(1) The cofinality of $\mathcal{C}$ is $\geq \kappa$.

(2) $\cup\mathcal{C}$ is dense in X.

(3) Given C, C' $\in \mathcal{C}$, cl C $\subsetneqq$ C' whenever C $\subsetneqq$ C'.

(4) Every subset $\mathcal{C}'$ of $\mathcal{C}$ of cofinality $\geq \kappa$ [resp. coninitiality $\geq \kappa$] has the property that $\mathrm{cl}(\cup\mathcal{C}') \supset \mathrm{int}\,(\cap\, u\, (\mathcal{C}')$ [resp. $\mathrm{int}\,(\cap\mathcal{C}') \subset \mathrm{cl}(\cup\, \ell\, (\mathcal{C}')]$.

The notation $u\ (\mathcal{C}')$ stands for $\{\mathcal{C} \in \mathcal{C}$: C' $\subset$ C for all C' $\in \mathcal{C}'\}$ while $\ell\ (\mathcal{C}')$ stands for $\{C \in \mathcal{C}$: C $\subset$ C' for all C' $\in \mathcal{C}'\}$.

Definition 2. Let X be a space. A *solid tunnel through* X is a chain $\mathcal{C}$ of open subsets of X with no greatest member, satisfying (2) and (3) of Definition 1 and (4+) for every $\mathcal{C}' \subset \mathcal{C}$, $\mathrm{cl}(\cup\mathcal{C}') \supset \mathrm{int}\,(\cap\, u\, (\mathcal{C}'))$.

Definition 3. A *2-way κ-tunnel through* X is a κ-tunnel $\mathcal{C}$ through X such that $\cap\mathcal{C}$ has empty interior. A *complete tunnel through* X is a solid tunnel $\mathcal{C}$ through X such that $\cap\mathcal{C}$ has empty interior.

Theorem 7. *The Complete Tunnel Axiom is equivalent to the statement that there is a complete tunnel through* $\beta\mathbf{N} - \mathbf{N}$. *In fact, Theorem* 7 *is true even if one substitutes any space* X *for* $\beta\mathbf{N} - \mathbf{N}$ *in both places.*

Despite its name, the ω_1-tunnel axiom does not appear to be equivalent to the statement that there is an ω_1-tunnel through $\beta\mathbf{N} - \mathbf{N}$; the latter statement follows from ZFC, but the former one "feels like" a ZFC-independent statement. However, the ω_1-tunnel axiom would follow from the statement that there is an ω_1-tunnel of *clopen* sets through $\beta\mathbf{N} - \mathbf{N}$, a statement implied by the Complete Tunnel Axiom.

An interesting sidelight is provided by:

Theorem 8. *Let* X *be a regular space. The following are equivalent.*

(1) There is <u>*no*</u> *simple increasing* ω*-tunnel* $\{C_n : n \in \omega\} C_n \subset C_{n+1}$ *through* X.

(2) X *is feebly compact and every nonempty* G_δ *set in* X *has nonempty interior.*

Tunnels through $\beta\mathbf{N} - \mathbf{N}$ are intimately related to tight near-gaps in $\mathcal{P}^*(\omega)$, the collection of all infinite, co-infinite subsets of ω. In what follows, $A < B$ means "A-B is finite and B-A is infinite" and $A \leq B$ means "either $A < B$ or $A = B$."

Definition 4. Let $\mathcal{C}$ be a $\leq$-chain in $\mathcal{P}^*(\omega)$, and let $\mathcal{C}'$ and $\mathcal{C}''$ be subsets of $\mathcal{C}$. Then $\langle \mathcal{C}', \mathcal{C}'' \rangle$ is a (κ, λ^*)*-near-gap* in $\mathcal{P}^*(\omega)$ if

(1) Every member of $\mathcal{C}'$ < every member of $\mathcal{C}''$.

(2) The cofinality of $\mathcal{C}'$ is κ and the coninitiality of $\mathcal{C}''$ is λ.

(3) There does not exist a pair A_1, A_2 of distinct subsets of ω such that $A_1 < A_2$, and A_1 > every member of $\mathcal{C}'$, A_2 > every member of $\mathcal{C}''$.

Definition 5. A near-gap $\langle \mathcal{C}', \mathcal{C}'' \rangle$ in $\mathcal{P}^*(\omega)$ is *tight* [resp.--*a gap*] if there is no infinite $A \subset \omega$ such that A < every member of $\mathcal{C}''$ and almost disjoint from every member of $\mathcal{C}'$ [resp. and > every member of $\mathcal{C}'$].

Every complete tunnel of clopen sets through $\beta\omega - \omega$ is associated with a chain $\mathcal{C}$ of sets in $\langle \mathcal{P}^*(\omega), \leq \rangle$ such that $\langle \mathcal{C}', u(\mathcal{C}') \rangle$ is a tight near-gap for all $\mathcal{C}' \subset \mathcal{C}$, and such that $\mathcal{C}$ is unbounded both above and below (for a solid tunnel, we require only "unbounded above").

Theorem 9. *The following are equivalent.*

1. $\neg P(\omega_2)$

2. There exists a tight $(\omega_1, \omega_1{}^*)$*-gap in* $\mathcal{P}^*(\omega)$.

3. The "Hausdorff gap" space [1] *can be made countably compact.*

Finally, here is a sequence of results, the last of which suggests that there may be a model of set theory in which every "nice" countably compact *normal* space is compact.

Theorem 10. *If* $P(\kappa^+)$, *then every separable countably compact space is "feebly initially* κ*-compact"; that is,* *every open cover by* $\leq\kappa$ *open sets has a finite subcollection whose closures cover the space.*

Theorem 11. $P(\omega_2)$ *is equivalent to "every separable countably compact space is feebly initially* ω_1*-compact."*

Theorem 12. *If* $P(\omega_2)$*, then every "nice" countably compact space which contains a copy of* ω_1 *is non-normal.*

Problem. Does there exist a model of set theory in which every first countable, countably compact, noncompact space contains a copy of ω_1?

Such a model can not satisfy the axiom $\clubsuit$.

References

[1] E. K. van Douwen, *Hausdorff gaps and a nice countably paracompact nonnormal space*, Topology Proceedings 1 (1976), 239-242.

[2] S. P. Franklin and M. Rajagopalan, *Some examples in topology*, AMS Transactions 155 (1971), 305-314.

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*Correction added in proof: the ω_1-tunnel axiom does not seem to be enough to give a "nice" countably compact non-normal space. One needs to add the following condition on the LOTS to the statement of the ω_1-tunnel axiom: every point which is the limit of a nontrivial sequence has a countable local base. It is not known whether this strengthening of the ω_1-tunnel axiom is implied by the Complete Tunnel Axiom.