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Research Announcement: A NOTE ON THE EQUI-WEAK TOPOLOGY

by

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A NOTE ON THE EQUI-WEAK TOPOLOGY

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In [1] it is shown that for a given family F of real-valued functions on a set X, there is a weakest topology on X for which F is equicontinuous. This is accomplished by associating with $F \subset R^X$ another family A(F) such that F is equicontinuous if and only if A(F) is continuous. Then the weak topology induced by A(F) is defined to be the *equi-weak* topology induced by F and is denoted by $\mathcal{J}^F$.

In the same paper, an analogy is noted between the role of equi-weak topologies in the theory of pseudometrizable spaces and the role of weak topologies in the theory of completely regular spaces. In particular, we have

Theorem 1. *A topology for* X *is pseudometrizable if and only if it is an equi-weak topology induced by a family of real-valued functions.*

The analogy is maintained if we require the functions to separate points and the topology to be Hausdorff.

Theorem 2. *A topology for* X *is metrizable if and only if it is an equi-weak topology induced by a point-separating family of real-valued functions.*

If we suppose that X is already endowed with a topology, we can carry the analogy one step further. For just as any subcollection of the continuous functions on X will induce (via the weak topology) a coarser completely regular topology, so any equicontinuous subcollection of C(X) will induce (via the equi-weak topology) a coarser pseudometrizable topology.

Theorem 3. *Let* $(X,\mathcal{J})$ *be a topological space and let* $F \subset C(X)$. *Then* $\mathcal{J}^F \subset \mathcal{J}$ *if and only if* F *is equicontinuous.*

Proof. Since F is equicontinuous in $\mathcal{J}^F$, it is also equicontinuous in any finer topology, $\mathcal{J}$ in particular. Conversely, F is equicontinuous only if A(F) is continuous. But $\mathcal{J}^F$ is the coarsest topology on X for which A(F) is continuous. Hence $\mathcal{J}^F \subset \mathcal{J}$.

In addition to its role in extending the analogy which was established previously, Theorem 3, in conjunction with Theorem 2, provides us with a strikingly simple characterization of submetrizability. A space is said to be *submetrizable* if its topology contains a metrizable subtopology.

Theorem 4. *A space* X *is submetrizable if and only if* C(X) *contains a point-separating equicontinuous subcollection.*

Proof. Let $\mathcal{J}_d$ be a metrizable subtopology of $\mathcal{J}$. By Theorem 2, $\mathcal{J}_d = \mathcal{J}^F$ for some point-separating family F. Then by Theorem 3, F is equicontinuous. Conversely, let $F \subset C(X)$ be equicontinuous and separate points. By Theorem 2, $\mathcal{J}^F$ is metrizable, and by Theorem 3, $\mathcal{J}^F \subset \mathcal{J}$.

References

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