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by

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Research Announcement

DECOMPOSITIONS OF S^3 WHICH ARE WEAKLY DEFINABLE BY CRUMPLED CUBES

Edythe P. Woodruff[1]

The results summarized here will be proved in "Decomposition spaces having arbitrarily small neighborhoods with 2-sphere boundaries II" [W3]. In the Conference talk and in this announcement (but not as presented in [W3]) the weakly definable concept is used.

Let G be an usc decomposition of S^3, H be the collection of nondegenerate elements of G, and P be the natural projection of S^3 onto S^3/G. A set $A \subset S^3$ is *saturated* if A is the union of elements of G. For a set $T \subset S^3$, let Sat T denote $\{p \in S^3$: p is a point in some $g \in G$ which intersects $T\}$. A *crumpled cube* is the closure of either component of the complement of a (possibly wild) 2-sphere in S^3.

In the standard terminology, a sequence of manifolds with boundary $\{M_i\}_{i\in\omega}$ in S^3 is called a *defining sequence* if

(1) $M_{i+1} \subset \text{Int } M_i$, and

(2) the set H = the set of components of $\bigcap_{i\in\omega} M_i$.

These conditions imply that each Bd M_i misses $C\ell$ H* (where H* is the union of the nondegenerate elements).

If each component of M_i is a 3-cell, then the decomposition is definable by 3-cells. Harrold [H] proved

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that if a decomposition G of S^3 is definable by 3-cells, then S^3/G is homeomorphic to S^3.

Notice that a decomposition is definable by crumpled cubes if and only if it is definable by 3-cells. This follows from the fact that each Bd M_i is disjoint from $C\ell$ H^*.

In this announcement we call a decomposition G of S^3 *locally definable by* K if for each $g \in H$ there is a sequence $\{K_i^g\}_{i \in \omega}$ of sets such that

(1) K is a compact set in S^3,

(2) K_i^g is homeomorphic to K,

(3) $K_{i+1}^g \subset \text{Int } K_i^g$,

(4) $g = \bigcap_{i \in \omega} K_i^g$, and

(5) Bd K_i^g misses H^*.

Locally definable by 3-cells and by crumpled cubes are not equivalent. Price [P] proved that if G is locally definable by 3-cells, then S^3/G is homeomorphic to S^3. Woodruff [W1] proved that if G is locally definable by crumpled cubes, then S^3/G is homeomorphic to S^3.

The present work is a further extension of these results.

We weaken the locally definable definition by replacing Bd C_i misses H^* by Bd C_i is saturated, and obtain:

Definition. A decomposition G of S^3 is called *weakly definable by crumpled cubes* if for each $g \in H$ there is a sequence $\{K_i^g\}_{i \in \omega}$ of crumpled cubes such that

(1) $K_{i+1}^g \subset \text{Int } K_i^g$,

(2) $g = \bigcap_{i \in \omega} K_i^g$, and

(3) Bd K_i^g is saturated.

We call $\{K_i^g\}_{i \in \omega}$ a *weak defining sequence at* g.

Conjecture. If a decomposition G is weakly definable by crumpled cubes, then S^3/G is homeomorphic to S^3.

The following special cases of this conjecture are proved in [W3].

Theorem 1. *If* G *is weakly definable by crumpled cubes and at each* $g \in G$ *each element of the weak defining sequence is bounded by a 2-sphere which is not wild on both sides, then* S^3/G *is homeomorphic to* S^3.

Theorem 2. *Suppose* G *is weakly definable by crumpled cubes; and at each* $g \in G$ *each element* F_i^g *of the weak defining sequence is bounded by a 2-shpere* S_i^g *which contains disjoint* 0*-dimensional* F_σ*-sets* F_1 *and* F_2 *such that* $F_1 \cup \text{Int } S_i^g$ *and* $F_2 \cup \text{Ext } S_i^g$ *are each* 1-ULC, *and* $P(F_1)$ *and* $P(F_2)$ *are disjoint* 0*-dimensional sets in* S^3/G. *Then* S^3/G *is homeomorphic to* S^3.

Recall that an usc decomposition is called *compact* 0*-dimensional* if $P(\text{C}\ell\ H^*)$ is a compact 0-dimensional set.

Corollary. Let G *be a compact* 0*-dimensional usc decomposition of* S^3. *Suppose* G *is weakly definable by crumpled cubes; and at each* $g \in G, F_i^g$ *of the weak defining sequence is bounded by a 2-sphere* S_i^g *which contains disjoint* 0*-dimensional* F_σ*-sets* F_1 *and* F_2 *such that*

$F_1 \cup \text{Int } S_i^g$ *and* $F_2 \cup \text{Ext } S_i^g$ *are each* 1-ULC, *and* $P(F_1)$ *and* $P(F_2)$ *are disjoint. Then* S^3/G *is homeomorphic to* S^3.

The proofs are extensions of methods in [W1].

Somewhat different hypotheses from those in the conjecture are used in the following theorem and corollaries.

Theorem 3. *Let* G *be an usc decomposition of* S^3. *Assume that for each* $g \in H$ *and open set* U *containing* g *there is an open set* X *such that* $g \subset X \subset U$; *the set* $P(X)$ *is a 2-sphere; and for the new usc decomposition* $G_g = \{g \in G\colon g \subset \text{Sat Bd } X\} \cup \{p \in S^3\colon p \notin \text{Sat Bd } X\}$ *of* S^3, *the decomposition space* S^3/G_g *is homeomorphic to* S^3. *Then* S^3/G *is homeomorphic to* S^3.

Notice that the hypotheses do not require that Bd X be a 2-sphere.

Corollary 3A. *Let* G *be an usc decomposition of* S^3. *Assume that for each* $g \in H$ *and open* U *containing* g *there is an open set* X *such that* $g \subset X \subset U$; *the set* $P(\text{Bd } X)$ *is a 2-sphere; each element of* Sat Bd X *is tame, and there are only countably many nondegenerate elements in* Bd X. *Then* S^3/G *is homeomorphic to* S^3.

Proof. This is immediate from the Starbird-Woodruff result [S,W].

Corollary 3B. *Let* G *be an usc decomposition of* S^3. *Assume that each* $g \in H$ *is a tame polyhedron and that* H *contains at most a countable number of arcs (but any* *number of other polyhedra). Also assume that for each* $g \in H$ *and open set* U *containing* g *there is a crumpled cube* X *such that* $g \subset \mathrm{Int}\ X \subset U$ *and* $\mathrm{Bd}\ X$ *is saturated. Then* S^3/G *is homeomorphic to* S^3.

Proof. This follows from Corollary 3A, since the conditions here imply that H_A is countable.

Remark. In Corollaries 3A and 3B the tameness condition can be replaced by assuming that each $g \in H$ has a mapping cylinder neighborhood in S^3. This follows from Theorem 2 in [W2].

Added in proof: Steve Armentrout has recently given a counterexample to the full conjecture.

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