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AN EXAMPLE CONCERNING LOTS WITH $\delta\theta$-BASES

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In 1966 Worrell and Wicke [WW] introduced the concept of a θ-base as a generalization of developability. In 1967 Bennett [B_1] introduced another generalization of developable spaces, namely, quasi-developable spaces. In 1971 Bennett and Lutzer [BL] showed that the concept of a θ-base and a quasi-developable space are the same. In 1974 Aull [A] introduced and studied spaces with $\delta\theta$-bases, an obvious generalization of spaces with θ-bases.

In [B_2] it was shown that a LOTS with a θ-base has a point-countable base and an example was given of a LOTS with a point-countable base that did not have a θ-base. In this note an example of a LOTS with a $\delta\theta$-base that does not have a point-countable base is given.

Let N, P and Q denote the set of natural numbers, the set of irrational numbers and the set of rational numbers respectively.

Definition 1. A base $\mathcal{B}$ for a topological space X is a *θ-base* (*$\delta\theta$-base*) for X if $\mathcal{B} = \cup\{\mathcal{B}_n | n \in N\}$ where if $x \in X$ and U is an open set containing x, then there exists $n(x) \in N$ such that x is in finitely (countably) many members of $\mathcal{B}_{n(x)}$ and there exists $B \in \mathcal{B}_{n(x)}$ such that $x \in B \subset U$.

Definition 2. A base $\mathcal{P}$ for a topological space X is a *point-countable base* if, for each $x \in X$, x is in at most countably many members of $\mathcal{P}$.

It is easy to see that if $\mathcal{P}$ is a point-countable base for a topological space X, then $\mathcal{P}$ is a $\delta\theta$-base for X.

Definition 3. A *linearly ordered topological space* (=LOTS) is a linearly ordered set equipped with the usual open interval topology of the given order. A subset C of a LOTS is *convex* (with respect to the given order $\leq$) if, whenever p and q are in C, then the closed interval between p and q is contained in C.

Example 1. There is a LOTS X with a $\delta\theta$-base that does not have a point-countable base.

Let $X = \{(x_1,x_2,\cdots,x_\lambda) \mid \lambda \leq \omega_1,\ x_\alpha \in P \text{ if } \alpha < \lambda,\ x_\lambda \in Q\}$. If x and y are in X and $x \neq y$, then there is a first ordinal γ such that $x_\gamma \neq y_\gamma$. If $x_\gamma \leq y_\gamma$ then let $x \leq y$. Equip X with the usual linear topology induced by $\leq$. If $x = (x_1,\cdots,x_\gamma)$, let $L(x) = \lambda$. We say that y extends x if $L(y) \geq L(x) = \lambda$ and $x_\alpha = y_\alpha$ for each $\alpha < \lambda$. If $x \in X$, $n \in N$ and $L(x) = \lambda$, let

$$U(n,x) = \{y \in X \mid y \text{ extends } x,\ x_\lambda - 1/n < y_\lambda < x_\lambda + 1/n\} \cup \{x\}.$$

It is not difficult to see that $\{U(n,x) \mid n \in N\}$ is a local base at x for each $x \in X$. If $x \in X$ and $L(x) < \omega_1$, then x extends at most countably many elements of X. Thus

$$\mathcal{U} = \{U(n,x) \mid x \in X, L(x) < \omega_1, n \in N\}$$

is a collection of open sets that is point-countable on $\{x \in X \mid L(x) < \omega_1\}$. Let $q_1,q_2,\cdots$ be a counting of Q and, if $(n,m) \in N^2$ let

$$\mathcal{V}(n,m) = \{U(n,x) \mid L(x) = \omega_1, x_{\omega_1} = q_m\}.$$

Notice if $x \neq y$, $L(x) = L(y) = \omega_1$ and $x_{\omega_1} = q_m = y_{\omega_1}$, then $U(n,y) \cap U(n,x) = \emptyset$ for each $n \in N$. Thus, if $\mathcal{V} = \cup\{\mathcal{V}(n,m) \mid (n,m) \in N^2\}$, it follows that $\mathcal{U} \cup \mathcal{V}$ is a $\delta\theta$-base for X.

To see that X does not have a point-countable base let $\mathcal{B}$ be any base for X. Choose $p_\alpha \in P$ $(1 \leq \alpha \leq \omega_1)$, and $x_\alpha \in X$, $n_\alpha \in N$, and $B_\alpha \in \mathcal{B}$ $(2 \leq \alpha \leq \omega_1)$ so that

$(i)_\alpha$ $x_\alpha = (p_1, p_2, \cdots, p_\beta, \cdots, 1)$ with $L(x_\alpha) = \alpha$;

$(ii)_\alpha$ $x_\alpha \in U(n_\alpha, x_\alpha) \subseteq B_\alpha \subseteq U(1, x_\alpha)$; and

$(iii)_\alpha$ if $2 \leq \beta \leq \alpha$, then $x_\alpha \in U(n_\beta, x_\beta)$,

for $2 \leq \alpha \leq \omega_1$. Conditions (i)-(iii) actually describe how the recursion is carried out. Thus, when α is a limit ordinal, $(i)_\alpha$ defines x_α automatically; $(iii)_\alpha$ follows from $(iii)_\beta$ for $\beta < \alpha$; and $(ii)_\alpha$ shows how to choose first B_α, then n_α. And when $\alpha = \gamma + 1$, choose p_γ so that x_α (as given by $(i)_\alpha$) is an element of $U(n_\gamma, x_\gamma)$; $(iii)_\alpha$ is then clearly true, and as before $(ii)_\alpha$ guides the choice of B_α and n_α.

By $(iii)_{\omega_1}$, $x_{\omega_1} \in \cap\{B_\alpha : 2 \leq \alpha \leq \omega_1\}$. But if $2 \leq \beta \leq \alpha \leq \omega_1$, then $x_\beta \notin U(1, x_\alpha) \supseteq B_\alpha$; since $x_\beta \in B_\beta$, it follows that $B_\alpha \neq B_\beta$. Thus, $\mathcal{B}$ is not point-countable (at x_{ω_1}), and X has no point-countable base.

Bibliography

[A] C. E. Aull, *Quasi-developments and δθ-bases*, J. London Math. Soc. 2 (1974), 197-204.

[B_1] H. R. Bennett, *Quasi-developable spaces*, Topology Conference, Arizona State University, 1967, Tempe, Arizona, 314-317.

[B_2] ________, *A note on point-countability in linearly* *ordered spaces*, Proc. Amer. Math. Soc. 28 (1971), 598-606.

[BL] _______ and D. J. Lutzer, *A note on weak θ-refinability*, Gen. Top. and Its Appl. 2 (1972), 49-54.

[WW] J. M. Worrell and H. H. Wicke, *Characterizations of developable spaces*, Canad. J. Math. 17 (1965), 820-850.