SURJECTIVE APPROXIMATE ABSOLUTE (NEIGHBORHOOD) RETRACTS

by

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In 1978 Paul R. Patten [P1] introduced the notions of a surjective approximate absolute retract (SAAR), a surjective approximate absolute neighborhood retract in the sense of Noguchi (SAANR$_N$), and a surjective approximate absolute neighborhood retract in the sense of Clapp (SAANR$_C$). He wanted to get generalizations of an absolute retract (AR) and an absolute neighborhood retract (ANR) which are locally connected and are therefore more natural than the original generalizations of Noguchi [N] and Clapp [C] if one wishes to study the properties of the image of a compact ANR under a certain class of maps (see [P2]).

The purpose of this note is to improve some results in [P1] and to prove some new results about Patten's notions.

Our results and presentation were greatly improved thanks to helpful comments by the referee.

We start by recalling the definitions of all generalizations of compact ANR's mentioned above. We assume that the reader is familiar with shape theory of compacta [B4].

We shall say that maps $f,g: Z \to Y$ of a space Z into a metric space (Y,d) are ε-*close* provided $d(f(z),g(z)) < \varepsilon$ for each $z \in Z$. If Z is a subset of Y and $f: Z \to Y$ is ε-close to the inclusion $i_{Z,Y}$ of Z into Y, then f is called an ε-*push*.

Definition. A map r: Y → X of a metric space Y into
its subspace X is an ε-*retraction* provided r|X is an ε-push.
If, in addition, r(X) = X, then r will be called a *surjec-
tive* ε-*retraction.* When there is an ε-retraction of Y onto
X for every ε > 0, we say that X is an *approximate retract*
of Y. X is a *surjective approximate retract* of Y if for
every ε > 0 there is a surjective ε-retraction of Y onto X.

Definition. By a *(surjective) approximate neighbor-
hood retract of a metric space Y in the sense of Noguchi*
will be meant a compact subspace X of Y which is a (sur-
jective) approximate retract of some neighborhood of X in Y.

Definition. By a *(surjective) approximate neighborhood re-
tract of a metric space Y in the sense of Clapp* will be meant a com-
pact subspace X of Y such that for every ε > 0 there is a neighbor-
hood U of X in Y and a (surjective) ε-retraction r: U → X.

If a compact metric space X is a (surjective) approxi-
mate retract [a (surjective) approximate neighborhood
retract in the sense of Noguchi or Clapp] of every metric
space in which it is embedded, then X is said to be a
(surjective) approximate absolute retract [a *(surjective)
approximate absolute neighborhood retract in the sense of
Noguchi or Clapp*]. These will be abbreviated as AAR
(SAAR), [AANR$_N$ (SAANR$_N$), or AANR$_C$ (SAANR$_C$)], respectively.

It was observed in [P1] that every SAAR, SAANR$_N$, or
SAANR$_C$ is locally connected (i.e., a Peano compactum) so
that the class of SAAR, SAANR$_N$, or SAANR$_C$ is properly con-
tained in the class of AAR, AANR$_N$, or AANR$_C$, respectively.

In the sequel we shall need the following proposition
resembling (2.1) in [C] (see also [P1, Lemma 2]).

Proposition 1. A compactum X *in the Hilbert cube* Q

is an SAAR (*an* SAANR$_N$ *or an* SAANR$_C$) *iff* X *is a surjective*

approximate (neighborhood) *retract of* Q (in the sense of

Noguchi or Clapp, respectively).

Proof. Suppose that a compactum X in Q is a surjec-

tive approximate retract of Q and consider X also as a sub-

set of a metric space M. Since every metric space can be

embedded as a closed subset of an AR [Hu, p. 81] without

loss of generality we can assume that M is an AR. Let

f: M → Q be an extension of the identity map id$_X$ and let

r: Q → X be a surjective ε-retraction. Then r ∘ f: M → X

is also a surjective ε-retraction. Hence, X is an SAAR.

The converse and the proofs of the remaining two state-

ments are obvious.

Corollary 1. Let X *and* Y *be compacta such that* X *is*

a surjective approximate retract of Y. *Then*

a) Y ∈ AR *implies* X ∈ SAAR; *and*

b) Y ∈ ANR *implies* X ∈ SAANR$_N$.

Proof. Consider X and Y as subsets of Q. The compo-

sition of a retraction of (a neighborhood N of Y in) Q and a

surjective ε-retraction of Y into X will be a surjective

ε-retraction of (the neighborhood N of X in) Q onto X.

Hence, Proposition 1 applies.

An interesting problem is to decide whether a surjec-

tive approximate retract X of an SAAR (SAANR$_C$ or SAANR$_N$)

Y is an SAAR (SAANR$_C$ or SAANR$_N$, respectively). The obvious

attempt is to compose a surjective (ε/2)-retraction r of

(a neighborhood N of Y in) Q onto Y and a surjective

($\varepsilon/2$)-retraction r_1 of Y onto X to get an ε-retraction of

(the neighborhood N of X in) Q onto X. However, it is not

clear that $r_1 \circ r(X) = X$. This will hold provided the

following statement is true.

(*) Let Y and X, $X \subset Y$, be locally connected compacta.

Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every

δ-push f: Y $\to$ Y of Y onto itself, there is an ε-push

g_0: f(X) $\to$ X of f(X) onto X which extends to a map g: Y $\to$ Y.

The next two theorems show that the question as to

whether an SAANR$_C$ X is either an SAAR or an SAANR$_N$ depends

only on shape properties of X. The first of them improves

Theorem 2 in [Pl] and resembles Theorem 7 in [Bo] while

the second is similar to Theorem 8 in [Bo].

Theorem 1. A compact metric space X is an SAAR *iff*

X *is both an* FAR *and an* SAANR$_C$.

Proof. Suppose X is an SAAR. Then X is also an AAR

and therefore it is an SAANR$_C$ and an FAR because Bogatyi

[Bo] showed that FAR + AANR$_C$ $\Leftrightarrow$ AAR.

Conversely, let X be both an FAR and an SAANR$_C$. We

shall consider X as a subset of the Hilbert cube Q and

prove that X is a surjective approximate retract of Q.

This will clearly suffice by Corollary 1.

For a given $\varepsilon > 0$, since X is an SAANR$_C$, there is a

neighborhood U of X in Q and a surjective ε-retraction

r: U $\to$ X. But, since X is also an FAR, by Borsuk's

characterization of compacta with trivial shape [B3,

Theorem (1.1)], there is a map $f: Q \to U$ such that $f|X = i_{X,U}$. The composition $R = r \circ f: Q \to X$ is a surjective ε-retraction of Q onto X. Hence, X is a surjective approximate retract of Q.

Theorem 2. *A compact metric space* X *is an* SAANR$_N$ *iff* X *is both an* FANR *and an* SAANR$_C$.

Proof. Since an SAANR$_N$ is clearly an AANR$_N$ and an AANR$_N$ is an FANR by [G], it follows that an SAANR$_N$ is both an FANR and an SAANR$_C$.

Conversely, suppose that X is an FANR and an SAANR$_C$. We shall assume that X is a subset of Q and prove that X is a surjective approximate neighborhood retract of Q in the sense of Noguchi. By Corollary 1, this would imply that X is an SAANR$_N$.

Since X is an FANR, there is a neighborhood V of X in Q such that for every neighborhood W of X in Q there is a homotopy $f_t^W: V \to Q$, $0 \leq t \leq 1$, satisfying $f_0^W = i_{V,Q}$, $f_1^W(V) \subset W$, and $f_1^W|X = i_{X,Q}$ [B4, p. 264]. On the other hand, since X is an SAANR$_C$, for each $\varepsilon > 0$ there is a neighborhood W_ε of X in Q and a surjective ε-retraction $r_\varepsilon: W_\varepsilon \to X$. But, the composition $r_\varepsilon \circ f_1^{W_\varepsilon}: V \to X$ is also a surjective ε-retraction. Hence, X is a surjective approximate neighborhood retract of Q in the sense of Noguchi.

Corollary 2. *Every one-dimensional* SAANR$_N$ *and every plane* SAANR$_N$ *is an* ANR.

Proof. Let X be a one-dimensional SAANR$_N$. Then X is

locally connected and its first Betti number is finite
because X is an FANR. Hence, X is an ANR [B2, p. 138].

Similarly, if X is a plane SAANR$_N$, then X is locally
connected and its complement $R^2 - X$ has a finite number of
components because X is an FANR. Hence, X is an ANR
[B2, p. 138].

Corollary 3. *Every one-dimensional* SAAR *and every
plane* SAAR *is an* AR.

Definition. Let C be a class of compacta. A compact
metric space X is said to be *quasi-C space* if for every
ε > 0 there exist a space K ∈ C, a map f from X onto K,
and a map g from K onto X such that g ∘ f is an ε-push.

Note that a compactum X is a quasi-ANR (a quasi-AR)
[P1] iff X is quasi-C space when C is the class ANR (AR) of
all compact ANR's (AR's).

Proposition 2. *Let C and D be classes of compacta
and let* X *be a quasi-C space.*

a) *If each member of C is an* SAAR, *then* X *is an* SAAR.

b) *If each member of C is an* SAANR$_C$, *then* X *is an*
SAANR$_C$.

c) *If each member of C is a quasi-D space, then* X *is
a quasi-D space.*

Proof of a). Without loss of generality we can assume
that X and every member of C are compacta in Q. For an
ε > 0 pick K ∈ C and maps f: X → K of X onto K and g: K → X
of K onto X such that g ∘ f is an (ε/2)-push. Choose a

$\delta > 0$ such that g maps δ-close points into $(\varepsilon/2)$-close

points.   Let F: Q → Q be an extension of f and let

r: Q → K be a surjective δ-retraction.   It is easy to

check that g ∘ r ∘ F is a surjective ε-retraction of Q onto

X.   Hence, by Proposition 1, X is an SAAR.

The proof of b) is similar to the above proof of a).

Proof of c).   First choose K ∈ $\mathcal{C}$ and maps f: X → K

of X onto K and g: K → X of K onto X with g ∘ f $(\varepsilon/2)$-close

to the id_X.   Then pick L ∈ $\mathcal{D}$ and maps α: K → L of K onto L

and β: L → K of L onto K such that β ∘ α is δ-close to id_K,

where $\delta > 0$ has the property that g maps δ-close points

of K into $(\varepsilon/2)$-close points of X.   Clearly, the diagram

$$X \xrightarrow{\ id_X\ } X$$
$$\alpha\circ f \searrow \qquad \nearrow g\circ\beta$$
$$L$$
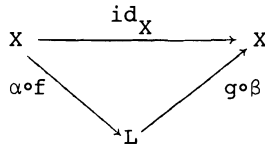
is ε-commutative.   Hence, X is a quasi-$\mathcal{D}$ space.

In the investigation of properties of $AANR_C$'s the

metric of continuity d_C [B1] played an important role [C].

For $SAANR_C$'s the metric $d_{\underline{C}}$ defined below has a similar

role.   It was introduced by Mazurkiewicz [M].   If A,B ∈ 2^Y

(i.e., A and B are nonempty compacta in a metric space

(Y,d)) and there exist surjections from A to B and from

B to A, then $d_{\underline{C}}(A,B)$ is the infimum of all $\varepsilon > 0$ for which

there are ε-pushes of A onto B and B onto A.

Corollary 4.   Let $\mathcal{C}$ be a class of compacta, let

$\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta in a metric space Y, and*

let $\lim d_{\underline{C}}(A_n,A_0) = 0$.   *If each A_n (n = 1,2,···) is a*

quasi-C space (an SAAR, an SAANR$_C$), *then* A_0 *is also a*

quasi-C space (an SAAR, an SAANR$_C$, *respectively*).

   Proof. Since A_0 is a quasi-$\{A_1, A_2, \cdots\}$ space, we can

apply Proposition 2.

   Proposition 3.   Let $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta*

in an ANR Y *and let* $\lim d_{\underset{C}{}}(A_n, A_0) = 0$. *If each* A_n

($n = 1, 2, \cdots$) *is a surjective approximate neighborhood*

retract of Y *in the sense of Clapp, then* A_0 *is also a sur-*

jective approximate neighborhood retract of Y *in the sense*

of Clapp.

   Proof. For an $\varepsilon > 0$, pick an index $n > 0$ such that

$d_{\underset{C}{}}(A_n, A_0) < \varepsilon/3$. Let u and v be ($\varepsilon/3$)-pushes of A_0 onto

A_n and A_n onto A_0, respectively. Since A_n is a surjective

approximate neighborhood retract of Y in the sense of

Clapp there is an open neighborhood V of A_n in Y for which

there is a surjective ($\varepsilon/3$)-retraction r: V $\to$ A_n. Let

u*: V* $\to$ V be an extension of the map u to a neighborhood

V* of A_0 in Y such that u* is also an ($\varepsilon/3$)-push. Then

v $\circ$ r $\circ$ u*: V* $\to$ A_0 is a surjective ε-retraction of V*

onto A_0.

   Corollary 5.   Let X *be a compact subset of* Q. *If*

there is a sequence $\{P_n\}$ *of polyhedra in* Q *such that*

X = $\lim P_n$ *in the metric* $d_{\underset{C}{}}$, *then* X *is both an* SAANR$_C$ *and*

a quasi-ANR.

   It is interesting that the converse of Corollary 5 is

also true. In order to prove that (in Theorem 3 below) we

shall need the following lemma which describes a useful

property of an SAANR$_C$. Since a compactum is an SAANR$_C$ iff
it has finitely many components each of which is an SAANR$_C$
it suffices to consider only a connected SAANR$_C$.

A map f: X → Y of a space X into a metric space (Y,d)
is ε-*dense* if for every y ∈ Y there is an x ∈ X with
d(f(x),y) < ε. We shall say that a Peano continuum Y is
an S-*space* provided for every ε > 0 there is a δ > 0 such
that every δ-dense map f: X → Y of a Peano continuum X
into Y is ε-close to a map f': X → Y of X onto Y.

Lemma 1. *Every connected* SAANR$_C$ *is an* S-*space.*

Proof. Let B be a connected nondegenerate SAANR$_C$
and let an ε > 0 be given. Assume that B is a subset of
Q and select a compact neighborhood U of B in Q for which
there is a surjective (ε/2)-retraction r: U → B. Pick the
required δ > 0 so that r maps 6δ-close points of U into
(ε/2)-close points of B and so that the open 6δ-neighbor-
hood N$_{6\delta}$(B) of B lies in U.

Consider a δ-dense map f: A → B of a Peano continuum
A into B. We can assume that the image f(A) is non-
degenerate (by taking a smaller δ if necessary). By Lemma
5 in [LM], there is a 6δ-push h: f(A) → Q such that
h(f(A)) ⊃ N$_{2\delta}$(f(A)). But, since f is a δ-dense map into B,
N$_\delta$(B) ⊂ N$_{2\delta}$(f(A)) so that h(f(A)) ⊃ N$_\delta$(B). This implies
that f' = r ∘ h ∘ f: A → B is a map of A onto B which is
ε-close to f.

Lemma 2. *Let* $\mathcal{C}$ *be a class of* S-*spaces. Then every
quasi-*$\mathcal{C}$ *continuum* X *is also an* S-*space.*

Proof. For a given ε > 0, pick an S-space K ∈ ζ, a map f of X onto K, and a map g of K onto X such that g ∘ f is an (ε/2)-push of X onto itself. Let δ_1 > 0 be such that g maps δ_1-close points into (ε/2)-close points. Since K is an S-space, there is an η > 0 with the property that every η-dense map of a Peano continuum into K is δ_1-close to a map onto K. Finally, the required δ > 0 is chosen so that f maps δ-close points into η-close points.

The equivalence of (i) and (ii) in the next theorem was first proved in [Pl, Theorem 1].

Theorem 3. *For a compact connected subset X of Q the following are equivalent.*

   (i) X is a quasi-ANR.

   (ii) X is an SAANR$_C$.

   (iii) X is an AANR$_C$ *and an S-space.*

   (iv) There is a sequence {P$_n$} *of connected polyhedra in Q such that X = lim P$_n$ in the metric* d$_{\bar{C}}$.

   Proof. Since (i) ⟹ (ii) by Proposition 2(b), (ii) ⟹ (iii) by Lemma 1, and (iv) ⟹ (i) by Corollary 5, it remains to see that (iii) ⟹ (iv). For that implication it suffices to show that given an ε > 0, there is a polyhedron P in Q such that d$_{\bar{C}}$(X,P) < ε. Let ε > 0 be given. Pick a δ,0 < δ < ε/4, with respect to ε/2 using the fact that X is an S-space. Then we take an η,0 < η < δ/2, such that there is a δ-push r: N$_η$(X) → X [C, Lemma 4.2]. Next we select (see [HW, p. 73]) a connected polyhedron P* and a map f: X → P* of X onto P* with the preimage of each

point in P* of diameter $< \eta$ (i.e., f is an η-map of X onto

P*). According to [K, Theorem 9, §41], there is an embed-

ding h: P* $\rightarrow$ Q such that the composition h $\circ$ f is an η-push.

Hence, P = h(P*) is a polyhedron in Q contained in N_η(X)

and r|P: P $\rightarrow$ X is a δ-dense map of P into X. By the choice

of δ, there is a map g: P $\rightarrow$ X of P onto X which is $(\epsilon/2)$-

close to r|P. But, since r is a δ-push, g is an ϵ-push of

P onto X. Hence, $d_{\bar{C}}$(X,P) $< \epsilon$.

Corollary 6. A continuum X is an SAAR *iff X is an*

AAR *and an S-space.*

Corollary 7. Let M be a separable metric space and

let SAANR$_C$(M) *denote the hyperspace of all connected*

SAANR$_C$*'s in M with the topology of* $d_{\bar{C}}$. *Then* SAANR$_C$(M)

is a separable metric space.

Proof. Since M can be emebeded into Q, SAANR$_C$(M) is

a subspace of SAANR$_C$(Q). But, SAANR$_C$(Q) is separable

because finite polyhedra in Q form a separable dense sub-

set (by Theorem 3).

That the polyhedra are dense is proved as follows

(see Borsuk, Theory of retracts). Let G denote a countable

dense subset of Homeo (Q), the group of all homeomorphisms

of Q onto itself. Let P_g denote all finite nondegenerate

polyhedra in Q that are geometrically realized in Q (con-

sidered as a standard subset of ℓ_2). Let RP$_g$ denote the

countable subset of P_g consisting of all polyhedra in P_g

all vertices of which have all coordinates rational. It

is easy to see that G(RP$_g$) = {g(p)|p $\in$ RP$_g$, g $\in$ G} is a

countable dense subset in P(Q), the collection of finite

nondegenerate polyhedra in Q, with the topology of $d_{\bar{c}}$.

   Corollary 8. Let X be a connected SAANR$_C$. *Then both*

the hyperspace 2^X *of all nonempty compacta in X and the*

hyperspace C(X) of all nonempty subcontinua of X metrized

by the Hausdorff metric are SAAR's.

   Proof. Modify Clapp's proof in section 6 of [C] of

a similar result for AANR$_C$'s by requiring that all maps

are onto.

   Recall that a compactum X is $\mathcal{C}$-*like*, where $\mathcal{C}$ is a

class of compacta, if for every $\varepsilon > 0$ there is an ε-map of

X onto some member of $\mathcal{C}$. If each member of a class $\mathcal{D}$ of

compacta is $\mathcal{C}$-like, then we say that the class $\mathcal{D}$ is $\mathcal{C}$-*like*.

   Proposition 4. Let $\mathcal{C}$ *and* $\mathcal{D}$ *be classes of compacta*

and assume that $\mathcal{D}$ *is* $\mathcal{C}$-*like. If an* SAANR$_C$ *X is a quasi-*$\mathcal{D}$

space then X is also a quasi-$\mathcal{C}$ *space.*

   Proof. For a given $\varepsilon > 0$, pick a K $\in$ $\mathcal{D}$, a map f: X $\to$ K

of X onto K, and a map g: K $\to$ X of K onto X such that g $\circ$ f

is an ($\varepsilon/2$)-push. Since X is a quasi-ANR by Theorem 3,

we can use Lemma 1 in [Pl] and get a $\delta > 0$ such that if u

is a δ-map of K onto an L $\in$ $\mathcal{C}$, then there is a map v of L

onto X such that g is ($\varepsilon/2$)-close to v $\circ$ u. It is easy to

check that v $\circ$ (u $\circ$ f) is an ε-push of X onto itself.

Hence, X is a quasi-$\mathcal{C}$ space.

   Proposition 5. Let $\mathcal{C}$ *be a class of continua. If a*

connected SAANR$_C$ *X in Q is* $\mathcal{C}$-*like, then X = lim K$_i$ in the*

metric $d_{\bar{c}}$ *where each K$_i$ is homeomorphic to some member of* $\mathcal{C}$.

Proof. The proof is almost identical to the proof
of (iii) ⇒ (iv) in Theorem 3.

Corollary 9. Let q - $A\mathcal{R}$ *denote the class of all
quasi-AR's. If an* SAANR$_C$ *is* (q - $A\mathcal{R}$)*-like, then it is a
quasi*-AR.

Proof. Combine Corollary 4 and Proposition 5. This
corollary improves Theorem 3 in [Pl].

Corollary 10. If f: X → Y *is a refinable map* [FR]
of an SAANR$_C$ X *onto a quasi-AR* Y, *then* X *is also a quasi*-AR.

Proof. We can apply Corollary 9 because X is Y-like
[FR].

The last two corollaries gave partial answers to Pat-
ten's question: is every SAAR a quasi-AR? The following
proposition also gives a sufficient condition on an SAAR
to be a quasi-AR.

Proposition 6. Let X ⊂ Q *be an* SAAR *and assume that
there is an equicontinuous family* $\{r_k: Q \to X_k\}_{k=1}^{\infty}$ *of retrac-
tions with* $X_k \subset N_{1/k}(X)$ *and* X ⊂ X_k *for each* k > 0. *Then*
X *is a quasi*-AR.

Proof. Let an ε > 0 be given. Let U be a compact
neighborhood of X in Q for which there is an (ε/2)-push
r: U → X satisfying r(X) = X. Then select a δ > 0 such
that r maps δ-close points of U into (ε/2)-close points of
X. Next we pick an η > 0 and an index k_0 such that
$N_{3\eta}(X)$ ⊂ U and r_k maps 3η-close points of Q into δ-close
points for all k ≧ k_0. Since we can assume that X is a
nondegenerate Peano continuum, according to Lemma 5 in [LM],

there is a 3η-push h: X → Q such that h(X) $\supset$ N$_\eta$(X). Let

k $\geqq$ k$_0$ be such that X$_k$ $\subset$ N$_\eta$(X). It is easy to check that

u = r$_k$ ∘ h is a map of X onto X$_k$ and v = r$|$X$_k$ is a map of

X$_k$ onto X and that v ∘ u is an ε-push of X onto itself.

Proposition 7. Let P = $\Pi_{k>0}$X$_k$ *be the Cartesian product*

of continua X$_k$. *Then*

a) P *is an* SAAR *is each* X$_k$ *is an* SAAR;

b) P *is an* SAANR$_C$ *if each* X$_k$ *is an* SAANR$_C$; *and*

c) P *is an* SAANR$_N$ *if each* X$_k$ *is an* SAANR$_N$ *and almost*

all X$_k$'*s are* SAAR'*s.*

Proof of b). It is clear that the product of finitely

many SAANR$_C$'s is an SAANR$_C$. The statement for countably

infinitely many factors follows from Proposition 2(b)

because P is a quasi-$\mathcal{C}$ space, where $\mathcal{C}$ = {X^1,X^2,⋯} is the

set of all finite products X^n = $\Pi_{k=1}^n$X$_k$. Indeed, for n > 0,

let f$_n$: P → X^{n+1} denote the projection and let g$_n$: X^{n+1} → P

be a map defined by g$_n$(x$_1$,⋯,x$_n$,x$_{n+1}$) = (x$_1$,x$_2$,⋯,x$_n$,

ϕ(x$_{n+1}$)$_{n+1}$,ϕ(x$_{n+1}$)$_{n+2}$,⋯) where ϕ: X$_{n+1}$ → $\Pi_{k>n}$X$_k$ is a map

of a Peano continuum X$_{n+1}$ onto a Peano continuum $\Pi_{k>n}$X$_k$.

The composition g$_n$ ∘ f$_n$ will be close to the id$_P$ provided

n is large enough.

Proof of a). Use b), Theorem 1, and [B4, p. 193].

Proof of c). Use b), Theorem 2, and [B4, p. 195].

Proposition 8. Let K(X) *denote the cone and* S(X) *the*

(unreduced) *suspension of a continuum* X. *Then*

a) S(X) *is an* SAAR *if* X *is an* SAAR;

b) S(X) *is an* SAANR$_C$ *if* X *is an* SAANR$_C$;

c) $S(X)$ is an $SAANR_N$ if X is an $SAANR_N$; and

d) $K(X)$ is an $SAAR$ if X is an $SAANR_C$.

Proof. of b). Consider X as a subset of Q. By Theorem 3, there is a sequence $\{P_n\}$ of connected polyhedra in Q such that $X = \lim P_n$ in the metric $d_{\bar{C}}$. But, $\{S(P_n)\}$ is a sequence of connected polyhedra in the Hilbert cube $S(Q)$ [Ch] and $S(X) = \lim S(P_n)$ in the metric $d_{\bar{C}}$. Hence, by Theorem 3 again, $S(X)$ is an $SAANR_C$.

We shall close our remarks by identifying four interesting closed subsets of the hyperspace 2^Y of a metric space Y topologized by the metric $d_{\bar{C}}$ which are not closed in $(2^Y, d_C)$.

The first of them is provided by the class of all quasi-contractible compacta in Y. Recall [P1] that a compactum X is *quasi-contractible* provided for every $\varepsilon > 0$ there is an ε-push of X onto itself which is null-homotopic. Using Borsuk's Theorem (1.1) in [B3], it is easy to see that every quasi-contractible compactum has trivial shape. It follows from Theorem 2 in [P1] and our Theorem 1 that an $SAANR_C$ of trivial shape must be quasi-contractible.

Proposition 9. Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of compacta in a metric space Y and assume that $\lim d_{\bar{C}}(A_n, A_0) = 0$. If each A_n $(n = 1, 2, \cdots)$ is quasi-contractible, then A_0 is also quasi-contractible.

Proof. Let an $\varepsilon > 0$ be given. Select an index $n > 0$ so that $d_{\bar{C}}(A_n, A_0) < \varepsilon/4$. Let $f\colon A_0 \to A_n$ be an $(\varepsilon/4)$-push of A_0 onto A_n and let $g\colon A_n \to A_0$ be an $(\varepsilon/4)$-push of A_n

onto A_0. Observe that the composition $g \circ f$ is an $(\varepsilon/2)$-
push of A_0 onto itself. Pick a $\delta > 0$ with the property
that g maps δ-close points of A_n into $(\varepsilon/2)$-close points
of A_0. Since A_n is quasi-contractible, there is an δ-push
ϕ of A_n onto itself which is null-homotopic. Then $g \circ \phi \circ f$
is a null-homotopic ε-push of A_0 onto itself. Hence, A_0
is quasi-contractible.

The collection of all quasi-contractible compacta in
Y is not closed in $(2^Y, d_c)$ because the closure of the
$\sin(1/x)$-curve is a limit of arcs in the metric d_c and the
closure of the $\sin(1/x)$-curve is a set of trivial shape
which is not quasi-contractible.

Remark. The similar result can be proved for every
property $\mathcal{P}$ of maps such that $\phi \in \mathcal{P}$ implies $g \circ \phi \circ f \in \mathcal{P}$.
The examples of such properties besides "to be null-homo-
topic" are "to have category $\leqq k$," "to have cocategory
$\leqq k$,"$\cdots$(see [Ga]).

The second closed subset of $(2^Y, d_{\bar{c}})$ is formed by all
compacta A in Y having the surjective fixed point property
(sfpp) (i.e., such that for every map f of A onto itself
there is a point $a \in A$ with $f(a) = a$). Clearly, a compactum
with the fpp also has the sfpp, but the converse does not
hold (consider a disjoint union of an arc and a point).

Proposition 10. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta
in a metric space Y and assume that* $\lim d_{\bar{c}}(A_n, A_0) = 0$. *If
each* A_n *(n = 1,2,$\cdots$) has the* sfpp, *then* A_0 *also has the*
sfpp.

Proof. Suppose u is a map of A_0 onto itself without
fixed points. We shall find an index n > 0 and construct
a map v of A_n onto itself without fixed points. This is
an obvious contradiction to the assumption about A_n's.

Since a map u does not have fixed points, there is an
$\varepsilon > 0$ such that $d(x,u(x)) \geqq 3\varepsilon$ for each $x \in A_0$. Pick an
index n > 0 for which $d_{\overline{c}}(A_n,A_0) < \varepsilon$. Let f be an ε-push
of A_0 onto A_n and let g be an ε-push of A_n onto A_0. Then
the composition $v = f \circ u \circ g$ is a map of A_n onto itself
without fixed points because $d(y,v(y)) \geqq \varepsilon$ for each $y \in A_n$.

The collection of all compacta in Y having the sfpp
does not form a closed subset in $(2^Y,d_c)$ because the dis-
joint union of an arc and the Cantor set (which does not
have the sfpp) is the limit of disjoint unions of an arc
and finitely many points (that have the sfpp) in the metric
of continuity.

Corollary 11. *Let* $\mathcal{C}$ *be a class of compacta with the*
sfpp. *If an* $SAANR_C$ *X is* $\mathcal{C}$*-like, then X has the* sfpp.

Proof. Combine Propositions 5 and 10.

It follows from the next proposition and the Lemma 3
below that the third closed subset of $(2^Y,d_{\overline{c}})$ provide all
compacta in Y which quasi-embed into some space X. Recall
that a compactum A is *quasi-embeddable* into a space X if
for every $\varepsilon > 0$ there is an ε-map f of A into X (i.e., the
diameter of each set $f^{-1}(x)$, $x \in f(A)$, is less than ε).

Lemma 3. *Let* $\mathcal{C}$ *be a class of compacta and suppose*
$A_0 = \lim A_n$ *in the topology of* $d_{\overline{c}}$. *If* A_n *is* $\mathcal{C}$*-like for*

n = 1,2,3,$\cdots$, *then* A_0 *is* C-*like.*

   Proof.   Obvious.

   Note that the analogue of Lemma 3 for the metric d_c is false, as the example of a sequence of arcs converging to a point in the topology of d_c shows (using C as the class whose only member is an arc).

   Proposition 11.   *Let* C *be a class of compacta quasi-embeddable into a space X.   If a compactum A is* C-*like, then A is also quasi-embeddable into X.*

   Proof.   Let an $\varepsilon > 0$ be given.   Pick a $K \in C$ and an ε-map f of A onto K.   Since K is compact, there is a $\delta > 0$ such that if Z is a subset of K with a diameter $< \delta$, then $f^{-1}(Z)$ has diameter $< \varepsilon$.   Now, use the fact that K is quasi-embeddable into X and take a δ-map g of K into X.   Clearly, g $\circ$ f is an ε-map of A into X.

   The collection of all S-spaces in a metric space Y is the fourth example of a closed set in $(2^Y, d_{\bar{c}})$.   This set is clearly not closed in $(2^Y, d_c)$ because a limit of Peano continua in the metric d_c need not be a Peano continuum.

   Corollary 12.   *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of continua in a metric space Y and assume that* $\lim d_{\bar{c}}(A_n, A_0) = 0$.   *If each* A_n (n = 1,2,$\cdots$) *is an S-space, then* A_0 *is also an S-space.*

   Proof.   Since A_0 is clearly a quasi-$\{A_1, A_2, \cdots\}$ space, we can apply Lemma 2.

Finally, let us observe that there is no evidence either in this note or in [P1] to suggest that the following conjecture is not true.

Conjecture. Every locally connected $AANR_C$ is an $SAANR_C$.

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