LONGITUDINAL SURGERY ON COMPOSITE KNOTS

by

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1. Introduction

It was 1962 when Lickorish in [5] demonstrated that all compact connected orientable 3-manifolds without boundary could be represented as a result of surgery on a link in S^3. Since then a great effort has been made to decide the Poincare conjecture by considering surgery on knots and links. But surprisingly little else is known about the relationship between specific types of knots and the characteristics of the manifolds that can be obtained by surgery on those knots. This paper will demonstrate a relationship between knots and the Heegaard genus of manifolds.

Let X be a point set. We shall use int(X) for the interior of X, cl(X) for the closure of X, and ∂X for the boundary of X. If K is a cube-with-knotted hole, the longitudinal curve of K will be the simple closed curve on ∂K, unique up to isotopy, which bounds an orientable surface in K. The meridional curve of K will be the simple closed curve on ∂K, unique up to isotopy, which is transverse to the longitudinal curve, and which bounds a disk in $cl(S^3 - K)$. The genus of a manifold is defined to be the minimal genus of a Heegaard splitting of the manifold. Let X be a polyhedron contained in the P.L. n-manifold M. $N(X) \subset M$ is called a regular neighborhood of X in M if $X \subset int \, N(X)$ and N(X) is an n-manifold which can be simplicially collapsed to X.

This paper deals with piecewise linear topology. As
such, all manifolds are considered to be simplicial and
maps to be piecewise linear. This will be assumed as
additional hypotheses throughout the paper.

2. Algebraic Lower Bounds

Let k be an (m,2) torus knot and let K be its associated
cube-with-knotted hole. We shall use the notation nk to
stand for the composition of n copies of k, and nK to
stand for the cube-with-knotted hole associated with nk.
Since $\pi_1(K)$ has the form $\{x_1, x_2 | r_2\}$, we can use Van Kampen's
Theorem to compute $\pi_1(nK)$ by attaching each component of
the composite in such a way that one of the generators
of that component is identified with x_1. The result is
that $\pi_1(nK) \approx \{x_1, x_2, \cdots, x_{n+1} | r_2, r_3, \cdots, r_{n+1}\}$ where

$$r_i = (x_1 x_i)^{\frac{m-1}{2}} x_1 (x_i^{-1} x_1^{-1})^{\frac{m-1}{2}} x_i^{-1} \text{ for } 2 \leq i \leq n + 1.$$

Goodrick in [4] was able to show that this is an n + 1
generator group.

Let M^3 be the 3-manifold obtained from nK by sewing a
solid torus T to ∂nK in such a way that the boundary of a
meridional disk of T is sewn to the longitudinal curve of
nK. We have that $\pi_1(M^3) \approx \{x_1, x_2, \cdots, x_{n+1} | r_2, r_3, \cdots r_{n+1}, \mathcal{L}\}$
where $\mathcal{L} = L_2 L_3 \cdots L_{n+1}$ and

$$L_i = x_i (x_1 x_i)^{\frac{m-3}{2}} x_1 (x_1 x_i)^{\frac{m-1}{2}} x_1^{2(1-m)} \text{ for}$$
$$2 \leq i \leq n + 1.$$

Proposition 2.1. $\pi_1(M^3)$ *is an* n + 1 *generator group.*

Proof. Longitudinal surgery on a knot yields a

3-manifold whose infinite cyclic covering has the same
$z[t,t^{-1}]$ module-structure as the infinite cyclic cover of
the knot complement. This follows directly from the method
of construction of such covers as demonstrated in [7].
Therefore all of the Alexander invariants of the longi-
tudinal surgery manifold and the knot complement are identi-
cal. Since Goodrick in [4] used the elementary ideals of
the Alexander matrix to prove that $\pi_1(nK)$ was an n + 1
generator group, it follows that $\pi_1(M^3)$ is an n + 1
generator group.

   Corollary 2.2. If g is the Heegaard genus of M^3 *then*
$g \geq n + 1$.

   Proof. Any Heegaard splitting of M^3 can be associated
with a natural presentation of $\pi_1(M^3)$ which has as many
generators as the genus of the splitting. But Proposition
2.1 shows that any presentation must have at least n + 1
generators.

3. Geometric Upper Bounds

   An upper bound to the Heegaard genus of any 3-manifold
obtained by surgery on a link can be found by using the
"folklore" concept of a tunnel number.

   Definition. $t \subset K$ is a *tunnel* if t is a 3-cell and
$t \cap \partial K$ is a pair of disjoint 2-cells.

   It is well established that if K is a cube-with-
knotted hole, then it is possible to remove a series of
tunnels from K and thereby convert K into a handlebody.

   Definition. If L is an unsplittable link, the *tunnel*

number of L, T(L), is the minimum number of disjoint

tunnels needed to convert $cl(S^3 - N(L))$ into a handlebody.

   Proposition 3.1. *Let* nk *be the composition of* n

copies of an (m,2) *torus knot and let* nK = $cl(S^3 - N(nk))$.

The tunnel number of nk *is* n.

   Proof. In [1] it was shown that any torus knot has

tunnel number 1. We can find a set of n 2-spheres

$\{S_1, S_2, \cdots, S_n\}$ such that $S_i \cap$ nk is a pair of points, and

one component C_i of $S^3 - S_i$ contains the i-th component

of nk. We then can remove a single tunnel t_i from $C_i \cap$ nK

with $t_i \cap S_i = \emptyset$ for $1 \leq i \leq$ n and obtain a handlebody of

genus n + 1.

   The traditional approach to building a 3-manifold of

high Heegaard genus is to take the Cartesian product of a

closed 2-manifold of high genus with a circle. A quick

glance at the homology of such a manifold shows that it

cannot be obtained by surgery on a knot. However, the next

theorem will demonstrate that manifolds of any Heegaard

genus can be generated by surgery on knots.

   Theorem 3.2. *Let* n $\geq$ 0 *be an integer. There exist*

infinitely many knots k *such that surgery on* k *will yield*

a manifold of Heegaard genus n.

   Proof. Trivial surgery on any knot will, of course,

yield S^3, a manifold of genus 0. Louise Moser in [6] showed

that if k is a torus knot of type (r,s) and (p,q) surgery

is performed with $|rsp + q|$ = 1, then the manifold obtained

is a lens space. It was shown in [1] that any other type
of surgery on a torus knot yields a 3-manifold of genus 2.

Now suppose $n \geq 3$ and $(n - 1)k$ is the composition of
$(n - 1)$ copies of an $(m,2)$ torus knot. We know by Corol-
lary 2.2 that longitudinal surgery on $(n - 1)k$ will yield
a manifold of Heegaard genus at least n. By Proposition
3.1 we know that we can find a set $\{t_1, t_2, \cdots, t_{n-1}\}$ of
pairwise disjoint tunnels that by removal will transform
$\mathrm{cl}(S^3 - N((n-1)k)) = (n - 1)K$ into the handlebody H.

If we perform surgery on $(n - 1)k$, we have that the
image, $h(N((n - 1)k))$, in the resulting surgery manifold
M^3, is still a solid torus. Thus $h(N((n - 1)k) \cup t_1 \cup$
$\cdots \cup t_{n-1}$ is a handlebody H' of genus n. But
$\mathrm{cl}(M^3 - H') = H$ since H is uneffected by the surgery on
$(n - 1)k$. Thus any manifold obtained by surgery on
$(n - 1)k$ has a Heegaard splitting of genus n.

It is readily seen that this concept of a tunnel
number can yield an upper bound on the Heegaard genus of
any manifold obtained by surgery on a link. This idea was
developed in [2], where it was shown that the tunnel number
of knots and links acts differently from other invariants,
such as the bridge number. Also, in the examples we have
seen thus far, this maximal Heegaard genus is obtained
when longitudinal surgery is performed on the knot.

Question. Does longitudinal surgery on k always
yield a manifold of maximal Heegaard genus among those that
can be formed by surgery on k?

The author would like to thank the referee for a suggestion which shortened the proof of Proposition 2.1.

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