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1. Introduction

A *closure operator on* X is a function $c: \mathcal{P}(X) \to \mathcal{P}(X)$. In the past 75 years a number of authors have used various combinations of the following axioms to define closure operators.

C_1: $c(\phi) = \phi$,

C_2: $c(A) \supset A$,

C_3: $c(A \cup B) = c(A) \cup c(B)$,

$C_{3'}$: $A \subset B \Rightarrow c(A) \subset c(B)$,

C_4: $c(c(A)) \subset c(A)$.

There are at least two possible interpretations of closure operators. Depending on which one is used one is led to a different choice of requirements.

The first approach is to think of closure as a *hull operator* with respect to a given family of sets $\mathcal{C}$ (such as the closed sets, for example). This leads to

$$c(A) = \cap[C: C \in \mathcal{C}\ C \supset A].$$

An operator so defined must satisfy C_2, $C_{3'}$, and C_4 but need not satisfy C_1 or C_3.

In the second approach $c(A)$ is defined in terms of the *cluster points of the set* A as

$$c(A) = A \cup [x: x \text{ is a cluster point of } A].$$

Then c satisfies C_2. What other axioms c satisfies would depend on the properties assigned to cluster points. C_1

and C_3 are usually among these. F. Riesz who in 1906 was the first to use this interpretation, has C_3 and a stronger version of C_1.

If one starts with a convergence structure with relatively few properties (see Section 3 below) and defines x to be a cluster point of A iff there exists a filter $\mathcal{J}$ converging to x such that $A \in \mathcal{J}$ then C_1, C_2 and C_3 are all satisfied.

Another approach that also leads fairly directly to closure operators of the second kind is to introduce some abstract concept of nearness (contiguity, proximity) between families of sets (finite families, pairs). Again F. Riesz in 1908 was the first to explore this path by defining a "Verkettung" between pairs of sets. He did not require

$$A \cap B \neq \phi \Rightarrow A \text{ is "near" } B.$$

Today one generally makes this assumption and thus obtains

$$c(A) = [x: [x] \text{ is "near" } A].$$

The assumptions C_1, C_2, C_3 follow from requirements usually imposed on near structures.

It thus becomes desirable to study structures (X,c), where c is only required to satisfy the axioms C_1, C_2, C_3. Such structures were called *closure spaces* by Čech (1966) and were investigated by him.

Since that time further results have been obtained and it is now clear that topological spaces do not constitute a natural boundary for the validity of theorems but that many results can be extended to closure spaces. We give a brief survey of recent investigations of weak idempotency, separation axioms, types of compactness, extensions (in particular principal extensions), and a correspondence between certain near structures and certain compactifications, among others. We shall give proofs only for those results which have not previously been published or submitted for publication.

There are two articles: Chattopadhyay and Thron [1977] and Chattopadhyay, Njåstad and Thron [submitted] which we shall cite frequently. We shall refer to the first as CT and the second as CNT.

2. Grills and Basic Properties of Closure Spaces

In our treatment of closure spaces grills play a key role. A family $\mathcal{G} \subset \mathcal{P}(X)$ is called a *grill* if

$$\phi \notin \mathcal{G},$$

$$B \supset A \in \mathcal{G} \Rightarrow B \in \mathcal{G},$$

$$A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G} \text{ or } B \in \mathcal{G}.$$

We use the following notation $\Gamma(X)$ is the set of all grills on X, $\Phi(X)$ is the collection of all filters on X, and $\Omega(X)$ is the set of all ultrafilters on X. For all $\mathcal{G} \in \Gamma(X)$ one defines

$$\mathcal{G}^{\dagger} = [\mathcal{U}\colon \mathcal{U} \in \Omega(X),\ \mathcal{U} \subset \mathcal{G}].$$

$\mathcal{G}^{\dagger}$ is thus a subset of $\Omega(X)$. The mapping d: $\Phi(X) \cup \Gamma(X) \to \Phi(X) \cup \Gamma(X)$ given by

$$d(\mathcal{H}) = [B\colon B \cap H \neq \phi\ \forall H \in \mathcal{H}]$$

provides a 1 - 1 mapping from $\Phi(X)$ to $\Gamma(X)$. The following simple relations hold.

$$d(\mathcal{F}) = \cup[\mathcal{U}\colon \mathcal{U} \supset \mathcal{F}] \quad \forall \mathcal{F} \in \Phi(X)$$

$$d(\mathcal{G}) = \cap[\mathcal{U}\colon \mathcal{U} \subset \mathcal{G}] \quad \forall \mathcal{G} \in \Gamma(X).$$

It is easily proved that every grill is the union of ultrafilters and every union of ultrafilters is a grill.

Now let (X,c) be a closure space, that is c satisfies

$$c(\phi) = \phi, \; C(A) \supset A, \; c(A \cup B) = c(A) \cup c(B).$$

Then the family

$$[A\colon x \in c(A)]$$

is a grill for every $x \in X$. It will be called the *adherence grill of the point* x with respect to c and will be denoted by $\mathcal{G}_c(x)$. Thus

$$\mathcal{G}_c(x) = [A\colon x \in c(A)] \in \Gamma(X), \; \forall x \in X.$$

Adherence grills are the duals of neighborhood filters under the mapping d. It is a consequence of C_2 that $\mathcal{U}(x) = [A\colon x \in A]$, the principal ultrafilter containing $[x]$, is always contained in $\mathcal{G}_c(x)$. That is

$$\mathcal{U}(x) \subset \mathcal{G}_c(x), \; \forall x \in X.$$

If $\gamma = [\mathcal{G}_x\colon x \in X]$ is a family of grills on X subject only to the condition $\mathcal{U}(x) \subset \mathcal{G}_c(x)$ then a closure operator c_γ is defined by

$$c_\gamma(A) = [x\colon A \in \mathcal{G}_x], \; \forall A \subset X.$$

One shows easily that

$$\mathcal{G}_{c_\gamma}(x) = \mathcal{G}_x, \; \forall x \in X.$$

In view of the duality between filters and grills one can define convergence for grills rather than filters. One is led to the folloiwng formulations:

A grill $\mathcal{G}$ *converges to* x in the closure space (X,c) iff $\mathcal{G} \subset \mathcal{G}_c(x)$.

The point x is a *cluster point* of $\mathcal{G}$ iff $\mathcal{G}^{\dagger} \cap \mathcal{G}_c^{\dagger}(x) \neq \phi$. This condition says that $\mathcal{G}$ and $\mathcal{G}_c(x)$ have an ultrafilter $\mathcal{U}$ in common. But then $\mathcal{F} = d(\mathcal{G})$ and $\mathcal{N}_x = d(\mathcal{G}_c(x))$ are both contained in the ultrafilter $\mathcal{U}$. Hence for every $F \in \mathcal{F}$ and $N_x \in \mathcal{N}_x$ we have $F \cap N_x \neq \phi$ and hence x is a cluster point of $\mathcal{F}$ in the usual sense. Clearly if $\mathcal{G}$ converges to x then x is a cluster point of $\mathcal{G}$. For a further discussion of our definition see Section 11.

3. Convergence Structures in Their Relation to Closure Spaces

D. C. Kent (1969) building on earlier work of Choquet and Fischer introduced the concept of a convergence function. This can be restated in terms of grills as follows. Let $q: \Gamma(X) \to \mathcal{P}(X)$ be a function satisfying

$$K_1: \mathcal{U}(x) \subset \mathcal{G} \Rightarrow x \in q(\mathcal{G})$$

and

$$K_2: \mathcal{G} \subset \mathcal{G}' \Rightarrow q(\mathcal{G}) \supset q(\mathcal{G}')$$

then q shall be called a *convergence function on* X. This is to be understood in the following way. The grill $\mathcal{G}$ converges to all points $x \in q(\mathcal{G})$. Of course $q(\mathcal{G})$ may be the null set for many grills in $\Gamma(X)$.

To make the transition from a convergence function q on X to a closure operator one is led to consider

$$\mathcal{G}^{(q)}(x) = \cup[\mathcal{G}: x \in q(\mathcal{G})]$$

and to require that these grills be the adherence grills for the closure operator c to be constructed. From what we said earlier it follows that one will indeed get a closure operator, say c_q. However, convergence in terms of this closure operator may not be the same as that determined by q. In particular, $\mathcal{G}^{(q)}(x)$ converges to x in (X,c_q) but may not do so with respect to q since it is quite possible that $x \notin q(\mathcal{G}^{(q)}(x))$. This can be remedied by adding to our two previous requirements the third one:

$$K_3\colon\ x \in q(\cup[\mathcal{G}\colon x \in q(\mathcal{G})]).$$

The monotonicity requirement then insures that

$$x \in q(\mathcal{G}) \ \ \forall \mathcal{G} \subset \mathcal{G}^{(q)}(x) = \mathcal{G}_{c_q}(x).$$

Convergence functions satisfying K_1, K_2, K_3 (in terms of filters) were called by Kent pretopologies. He showed that a pretopology is equivalent to a closure structure in the sense of Čech, as we have just done. In view of this we prefer to call a convergence function satisfying K_1, K_2, K_3 a *closure equivalent* or ce-convergence function.

In the introduction it was suggested that (translated into grills and convergence functions)

$$x \in c_q(A) \Leftrightarrow \exists \mathcal{G} \in \Gamma(X),\ A \in \mathcal{G},\ x \in q(\mathcal{G}).$$

This can now be seen as follows: $x \in c_q(A)$ is equivalent to $A \in \mathcal{G}_{c_q}(x) = \mathcal{G}^{(q)}(x)$. For $A \in \mathcal{G}$, $x \in q(\mathcal{G})$ we have the equivalent formulation $A \in \mathcal{G} \subset \mathcal{G}^{(q)}(x)$, and so we see that the two statements are indeed equivalent.

We conclude this section by observing that the conditions K_1, K_2, K_3 are equivalent to

$$x \in q(\mathcal{U}(x)),\ \forall x \in X,$$

and

$$q(\mathcal{G}) = \cap[q(\mathcal{U})\colon \mathcal{U} \in \Omega(X),\ \mathcal{U} \subset \mathcal{G}],$$

for all $\mathcal{G} \in \Gamma(X)$, from which it also follows that a ce-convergence function q is completely determined once its values on $\Omega(X)$ are known. What we have proved can be pulled together into a theorem.

Theorem 3.1. *Let* $q: \Gamma(X) \to \mathcal{P}(X)$ *be a convergence function. Then the following statements are equivalent:*

(a) q *satisfies* K_1, K_2, K_3.

(b) q *satisfies*

$$x \in q(\mathcal{U}(x)),\ \forall x \in X,$$

$$q(\mathcal{G}) = \cap[q(\mathcal{U}): \mathcal{U} \in \Omega(X),\ \mathcal{U} \subset \mathcal{G}],\ \forall \mathcal{G} \in \Gamma(X).$$

(c) $c_q(A) = [x: \exists \mathcal{G} \in \Gamma(X),\ A \in \mathcal{G},\ x \in q(\mathcal{G})]$, $A \subset X$, *is a closure operator on* X.

4. Weakly Idempotent Closure Operators

Closure spaces differ from topological spaces in that their closure operators do not need to satisfy the idempotency requirement $c(c(A)) = c(A)$. Nevertheless, in some closure spaces, weaker forms of the idempotency axiom hold. A number of these can be subsumed under the following definition. For a fixed X and $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{P}(X)$ we shall mean by $\mathcal{A}/\mathcal{B}$ the collection of all closure spaces (X,c) for which

$$A \in \mathcal{A},\ B \in \mathcal{B},\ A \subset c(B) \Rightarrow c(A) \subset c(B).$$

If $(X,c) \in \mathcal{P}(X)/\mathcal{P}(X)$ then it is a topological space.

Of all possible classes of $\mathcal{A}/\mathcal{B}$ only two types have been studied to some extent. These are $\mathcal{P}(X)/\mathcal{A}$ and $\mathcal{A}/\mathcal{P}(X)$.

For the study of $\mathcal{P}(X)/\mathcal{A}$ spaces it is convenient to define

$$I(c) = [A: c(A) = c(c(A))].$$

$I(c)$ then consists of exactly those subsets of X for which the operator c is idempotent. We then have

Theorem 4.1. $(X,c) \in \mathcal{P}(X)/\mathcal{A}$ *iff* $\mathcal{A} \subset I(c)$.

Proof. If $(X,c) \in \mathcal{P}(X)/\mathcal{A}$ let $A \in \mathcal{A}$ then $c(A) \subset c(A) \Rightarrow c(c(A)) \subset c(A) \Rightarrow A \in I(c) \Rightarrow \mathcal{A} \subset I(c)$. If $\mathcal{A} \subset I(c)$ let $B \subset c(A)$, $A \in \mathcal{A}$. Then $c(B) \subset c(c(A))$ since C_3 implies that c is monotone. Now $A \in I(c)$ and hence $c(c(A)) = c(A)$ and $c(B) \subset c(A)$. It follows that $(X,c) \in \mathcal{P}(X)/\mathcal{A}$.

To investigate spaces in $\mathcal{A}/\mathcal{P}(X)$ it is helpful to introduce for all $\mathcal{G} \in \Gamma(X)$

$$D^{\mathcal{A}}(\mathcal{G}) = [B\colon \exists A \in \mathcal{A} \cap \mathcal{G} \text{ such that } c(B) \supset A].$$

In terms of the operator $D^{\mathcal{A}}$ we can characterize $\mathcal{A}/\mathcal{P}(X)$ as follows.

Theorem 4.2. $(X,c) \in \mathcal{A}/\mathcal{P}(X)$ *iff*

$$D^{\mathcal{A}}(\mathcal{G}_c(x)) \subset \mathcal{G}_c(x),\ \forall x \in X.$$

Proof. Assume $D^{\mathcal{A}}(\mathcal{G}_c(x)) \subset \mathcal{G}_c(x)$ for all $x \in X$. Let $A \in \mathcal{A}$ and let $A \subset c(B)$. Let $x \in c(A)$ then $A \in \mathcal{G}_c(x)$. Since $c(B) \supset A \in \mathcal{A} \cap \mathcal{G}_c(x)$ it follows that

$$B \in D^{\mathcal{A}}(\mathcal{G}_c(x)) \subset \mathcal{G}_c(x).$$

Hence $x \in c(B)$ and $c(A) \subset c(B)$ so that $(X,c) \in \mathcal{A}/\mathcal{P}(X)$.

If $(X,c) \in \mathcal{A}/\mathcal{P}(X)$ let B and x be such that $B \in D^{\mathcal{A}}(\mathcal{G}_c(x))$. Then there exists an $A \in \mathcal{A} \cap \mathcal{G}_c(x)$ such that $A \subset c(B)$. Since $A \in \mathcal{G}_c(x)$ we have $x \in c(A)$. From the $\mathcal{A}/\mathcal{P}(X)$ property it follows that $x \in c(A) \subset c(B)$. Hence $B \in \mathcal{G}_c(x)$ or $D^{\mathcal{A}}(\mathcal{G}_c(x)) \subset \mathcal{G}_c(x)$.

If $\mathcal{A} = \mathcal{P}(X)$ one obtains

$$D(\mathcal{G}) = D^{\mathcal{P}(X)}(\mathcal{G}) = [B\colon c(B) \in \mathcal{G}].$$

A grill satisfying the condition $D(\mathcal{G}) \subset \mathcal{G}$ is known as a c-grill. We then have as a corollary of Theorem 4.2.

Corollary 4.1. *A space* (X,c) *is topological iff* $\mathcal{G}_c(x)$ *is a* c-*grill for all* $x \in X$.

The corollary can be found in CT.

5. Separation Axioms

Parallel to the T_i-axioms, in their formulations but not necessarily in their properties, are the D_i-axioms.

D_0: $x \in c([y])$, $y \in c([x]) \Rightarrow x = y$.

D_1: $c([x]) = [x]$, $\forall x \in X$.

D_2: $x \neq y \Rightarrow \mathcal{G}_c^{\dagger}(x) \cap \mathcal{G}_c^{\dagger}(y) = \phi$.

A set A in a closure space (X,c) will be called *closed* iff $A = c(A)$. Regularity and complete regularity are stated for topological spaces in terms of closed sets. However in that setting sets are closed iff they are of the form $c(A)$. For our purposes it appears to be preferable to state the axioms in terms of closures of sets rather than in terms of closed set. We are thus led to the following statements:

Regular: $x \notin c(A) \Rightarrow \exists C,\ D \subset X,\ C \cap D = \phi$ such that $x \notin c(X \sim C)$, $A \cap c(X \sim D) = \phi$.

Completely regular: $x \notin c(A) \Rightarrow \exists f_{x,A}\colon (X,c) \to \mathbf{R}$ where $f_{x,A}$ is continuous and $f_{x,A}(x) = 0$, $f_{x,A}(c(A)) = 1$. Čech [1966] pointed out that regular closure spaces need not be topological but that completely regular closure spaces must be topological.

Diesto [1977] has studied these as well as other separation axioms. He also made a detailed study of various forms of connectedness for closure spaces. With few exceptions connectedness properties are completely determined by the closed sets of (X,c), that is they are topological properties of the topology induced by c.

Another approach to separation is to study the distinctness of adherence grills of different points. This leads to: $x \neq y \Rightarrow$

$$G_0\colon\ \mathcal{G}_c^{\dagger}(x) \neq \mathcal{G}_c^{\dagger}(y),$$
$$G_1\colon\ \mathcal{G}_c^{\dagger}(x) \not\subseteq \mathcal{G}_c^{\dagger}(y),$$
$$G_2\colon\ \mathcal{G}_c^{\dagger}(x) \cap \mathcal{G}_c^{\dagger}(y) = \phi.$$

The first two axioms could also have been stated in terms of $\mathcal{G}_c(x)$, $\mathcal{G}_c(y)$. This is not true for G_2, which of course is equal to D_2. The G_0-axiom was first used in CT. In general closure spaces the G_0-axiom is not equivalent to D_0. The D_0-axiom is the stronger of the two.

Theorem 5.1. $D_0 \Rightarrow G_0$.

Proof. We show that D_0 implies the contrapositive of G_0. Let $\mathcal{G}_c(y) = \mathcal{G}_c(x)$ then $x \in c([y])$ and $y \in c([x])$ and hence by D_0 $x = y$.

At this point it is convenient to introduce the following axiom

$$H_2\colon\ x \in c(A) \Rightarrow c([x]) \subset c(A).$$

Clearly this is the weak idempotency condition $\mathfrak{G}i(X)/\mathcal{P}(X)$, where $\mathfrak{G}i(X) = [[x]: x \in X]$. In H_2-closure spaces the axioms D_0 and G_0 are equivalent since it is true that

Theorem 5.2. $H_2 \cap G_0 \Rightarrow D_0$.

Proof. Let $y \in c([x])$ then $B \subset \mathcal{G}_c(x) \Rightarrow x \in c(B) \Rightarrow c([x]) \subset c(B) \Rightarrow y \in c(B) \Rightarrow B \subset \mathcal{G}_c(y) \Rightarrow \mathcal{G}_c(x) \subset \mathcal{G}_c(y)$.

Similarly $x \in c([y]) \Rightarrow \mathcal{G}_c(y) \subset \mathcal{G}_c(x)$.

Another axiom, though not properly a separation axiom is the *symmetry axiom*

$$S: x \in c([y]) \Rightarrow y \in c([x]).$$

In topological spaces the axiom S is frequently denoted by R_0. In terms of H_2 and S we have

Theorem 5.3. 1) $D_1 = G_1 \cap H_2$. 2) $D_1 = G_0 \cap S \cap H_2$.

Proof. Clearly $D_1 \Rightarrow H_2$ and $D_1 \Rightarrow S$. $D_1 \Rightarrow G_1$ since $\mathcal{G}_c(y) \subset \mathcal{G}_c(x) \Rightarrow x \in c([y])$. We now show $G_1 \cap H_2 \Rightarrow D_1$. Let $x \neq y$. Assume $x \in c([y])$. If $A \in \mathcal{G}_c(y)$ then $y \in c(A)$. Hence $c([y]) \in c(A)$. But then $x \in c(A)$ or $\mathcal{G}_c(y) \subset \mathcal{G}_c(x)$ from which $x = y$ follows. This is a contradiction and hence $c([y]) = [y]$ for all $y \in X$. Finally, we show $G_0 \cap H_2 \cap S \Rightarrow D_1$. Assume $y \in c([x])$. Let $A \subset \mathcal{G}_c(x)$ then $x \in c(A)$ so that $c([x]) \in c(A)$. Hence $y \in c(A)$ so that $\mathcal{G}_c(y) \subset \mathcal{G}_c(x)$. By S we also have $x \in c(y)$ and hence $\mathcal{G}_c(x) = \mathcal{G}_c(y)$ so that $y = x$ follows from G_0.

Thron and Warren [1973] studied S-closure spaces (they still used the notation "R_0"). Among their results the following may be of interest:

Every S-closure space is homeomorphic to a subspace of a product of spaces (X^*, c^*) where $X^* = [r,s,t]$ and

$$c^*([r]) = X^*,\ c^*([s]) = [r,s],\ c^*([t]) = [r,t].$$

Possibly the earliest separation axiom was suggested by F. Riesz [1908]. He did not deal with a closure operator but with a derived set operator, say v. Set

$$\mathcal{H}_v(x) = [A: x \in v(A)].$$

Riesz' axioms insure that $\mathcal{H}_v(x)$ is a grill for all $x \in X$. In addition one of his axioms states that for finite sets F $v(F) = \phi$. Hence $\mathcal{H}_v(x)$ consists only of non principal ultrafilters in his axiom system.

Riesz' separation axiom is

$$R: \mathcal{H}_v(x) = \mathcal{H}_v(y) \Rightarrow x = y.$$

If c and v are related by $c(A) = A \cup v(A)$ then

$$\mathcal{G}_c(x) = \mathcal{H}_v(x) \cup \mathcal{U}(x).$$

Thus all of the spaces considered by Riesz satisfy our G_0-axiom regardless of whether they satisfy the R-axiom. If the axiom R is imposed on a general closure space then the space need not be G_0. (Example: $\mathcal{H}_v(y) = \mathcal{U}(x)$, $\mathcal{H}_v(x) = \mathcal{U}(y)$.) It follows that, even though the two axioms are similar in structure, neither contains the other.

6. Compact Spaces

For topological spaces compactness can be expressed in a number of different ways. However for closure spaces some of these statements are not equivalent. With Čech (who formulated it in terms of nets) we choose as the definition of compactness for closure spaces the property:

every grill has a cluster point.

As a matter of fact we prefer the equivalent statement:

(X,c) is *compact* iff $[\mathcal{G}_c^\dagger(x): x \in X]$ is a cover of $\Omega(X)$.

An analogue of an "open cover" in topological spaces is provided by: $[A_i: i \in I]$ is a c-*cover* of X if $[X \sim c(X \sim A_i): i \in I]$ covers X. In his thesis Diesto raised the question whether *cover compactness* (every c-cover has a finite c-subcover) is equivalent to compactness. Before resolving this question we make a number of additional definitions. A grill $\mathcal{G}$ is called a *linked grill* if

$$A,B \in \mathcal{G} \Rightarrow c(A) \cap c(B) \neq \phi.$$

The grill is called F-*linked* if

$$A_1,\cdots,A_n \in \mathcal{G} \Rightarrow \cap_{k=1}^{n}[c(A_k)] \neq \phi.$$

In terms of this we say: a closure space (X,c) is called *linkage* (F-*linkage*) *compact* if every linked (F-linked) grill on X converges. As a first result we have the following.

Theorem 6.1. *The following statements for a closure space* (X,c) *are equivalent:*

(a) *If* $\mathcal{A} \subset \mathcal{P}(X)$ *satisfies* $\cap[c(A_k): A_k \in \mathcal{A}, k = 1,\cdots,n] \neq \phi$ *for all finite subsets of* $\mathcal{A}$ *then* $\cap[c(A): A \in \mathcal{A}] \neq \phi$.

(b) *If* $\mathcal{B} \subset \mathcal{P}(X)$ *is a* c-*cover of* X *then it has a finite* c-*subcover.*

(c) *Every* F-*linked grill on* X *converges.*

Proof. That (a) is equivalent to (b) is proved in the standard way (as for topological spaces). That (a) is equivalent to (c) was shown in CNT.

It thus follows that F-linkage compactness and cover compactness are identical. In CNT it was shown that there are F-linkage compact spaces which are not linkage compact and compact spaces which are not F-linkage compact (thus answering Diesto's question). However the following holds

Theorem 6.2. *Every linkage compact space is* F-*linkage compact. Every* F-*linkage compact space is compact.*

We note that for T_2-topological spaces linkage compact = F-linkage compact = compact. What the situation is for D_2-closure spaces is not known. It is equally an open question whether regular compact (F-linkage compact, linkage compact) closure spaces are topological. Finally, it is not known whether the various kinds of compactness have an influence on the degree of idempotency of the space.

7. Extensions of Closure Spaces

A more detailed account of extension theory on closure spaces can be found in CT. Here we review the basic terminology and give two new results concerning the interplay between extensions and weak idempotency.

An extension $E = (\Psi, (Y,k))$ of the closure space (X,c) is a pair where (Y,k) is a closure space and Ψ: $(X,c) \to (\Psi(X), k_{\Psi(X)})$ is a homeomorphism. We also require that $\Psi(X)$ be dense in (Y,k). One of the problems in the theory of extensions is to determine to what extent E is characterized by its trace on (X,c). This idea is made more precise by the following definitions.

$$\tau(y) = \tau(y,E) = [A: y \in k(\Psi(A))]$$

is called the *trace* of E at $y \in Y$. The family

$$X^* = X^*(E) = [\tau(y,E): y \in Y]$$

is called the *trace system* of the extension E. Two extensions E_1 and E_2 are called equivalent if there exists a homeomorphism χ: $(Y_1,k_1) \to (Y_2,k_2)$ such that on X $\chi \circ \Psi_1 = \Psi_2$. Note that $\tau(y) \in \Gamma(X)$, for all $y \in Y$, and that $\tau(\Psi(x)) = \mathcal{G}_c(x)$, $\forall x \in X$, so that $X^* \supset [\mathcal{G}_c(x): x \in X]$.

In the two theorems below properties of the traces of an extension are related to degrees of idempotency of the extension.

Theorem 7.1. *Let* $(\Psi,(Y,k))$ *be an extension of* (X,c) *and* $\mathcal{A} \subset \mathcal{P}(X)$. *If* (Y,k) *is a* $\Psi(\mathcal{A})/\mathcal{P}(\Psi(X))$ *space then all traces* $\tau(y)$, $y \in Y$ *satisfy*

$$D^{\mathcal{A}}(\tau(y)) \subset \tau(y).$$

Proof. $D^{\mathcal{A}}(\tau(y)) = [B\colon c(B) \supset A,\ A \in \mathcal{A},\ y \in k(\Psi(A))]$. Let $B \in D^{\mathcal{A}}(\tau(y))$ then $c(B) \supset A$ hence $\Psi(c(B)) \supset \Psi(A)$ and

$$k(\Psi(B)) \supset k(\Psi(B)) \cap \Psi(X) = \Psi(c(B)) \supset \Psi(A).$$

Since $(Y,k) \in \Psi(\mathcal{A})/\mathcal{P}(\Psi(X))$ it follows from $\Psi(A) \subset k(\Psi(B))$ that

$$k(\Psi(A)) \subset k(\Psi(B)).$$

Thus $y \in k(\Psi(B))$ and $B \in \tau(y)$.

Since $\mathcal{A}/\mathcal{B} \Rightarrow \mathcal{A}/\mathcal{B}'$ if $\mathcal{B}' \subset \mathcal{B}$ we have as a corollary of Theorem 7.1 that if (Y,k) is topological then all $\tau(y)$, $y \in Y$, are c-grills.

Theorem 7.2. *If* $(\Psi,(Y,k))$ *is an extension of* (X,c), *if* $\mathcal{A} \subset \mathcal{P}(X)$, *and if*

$$D^{\mathcal{A}}(\tau(y)) \subset \tau(y),\ \forall y \in Y,$$

then (Y,k) *is a* $\Psi(\mathcal{A})/\mathcal{P}(\Psi(X))$ *space.*

Proof. Let $A \in \mathcal{A}$, $\Psi(A) \subset k(\Psi(C))$ and $y \in k(\Psi(A))$. Note that $B \in D^{\mathcal{A}}(\tau(y))$ iff $\exists A' \in \mathcal{A}$, $y \in k(\Psi(A'))$ such that $c(B) \supset A'$. But then

$$k(\Psi(B)) \cap \Psi(X) = \Psi(c(B)) \supset \Psi(A')$$

or

$$k(\Psi(B)) \supset \Psi(A').$$

It follows that $C \in D^{\mathcal{A}}(\tau(y))$. Hence $C \in \tau(y)$ or $y \in k(\Psi(C))$. Thus, finally, $k(\Psi(A) \subset k(\Psi(C))$.

The conclusion of the theorem can be strengthened if more is known about k. Thus, for example if $k(Y \sim \Psi(X)) = Y \sim \Psi(X)$ then (Y,k) is a $\Psi(\mathcal{A})/\mathcal{P}(Y)$ space.

To simplify the theory one usually makes two assumptions:

1) $\tau(y)$ is a 1-1 function from Y to X^*,

2) (X,c) is a G_0-space.

We observe that 2) is necessary for 1). However, even the assumption that (Y,k) is a D_1-space is not sufficient for 1) to hold. Given 1) and 2), all extension, up to equivalence, can be obtained by setting $Y = X^*$,

$$\Psi(x) = \varphi(x) = \mathcal{G}_c(x) \in X^*, \; x \in X,$$

and

$$k(\alpha) = h_r(\alpha) = (\varphi^{-1}(\alpha))^* \cup r(\alpha \sim \varphi(X)), \; \alpha \in X^*,$$

where

$$A^* = [\mathcal{G}: \mathcal{G} \in X^*, \; A \in \mathcal{G}], \; A \subset X$$

and

$$r: \mathcal{P}(X^* \sim \varphi(X)) \to \mathcal{P}(X^*) \text{ satisfies}$$

$$r(\phi) = \phi, \; r(\alpha) \supset \alpha, \; r(\alpha \cup \beta) = r(\alpha) \cup r(\beta).$$

The choice of h_r insures that

$$h_r(\varphi(A)) = A^*$$

so that

$$\tau(\mathcal{G}) = [A: \mathcal{G} \in h_r(\varphi(A))] = [A: \mathcal{G} \in A^*] = \mathcal{G}.$$

From now on we shall use the phrase "trace system X^*" to mean "$\tau(\mathcal{G}) = \mathcal{G}, \; \forall \mathcal{G} \in X^*$" not just "$\tau(\mathcal{G}) \in X^*, \; \forall \mathcal{G} \in X^*$."

We thus have the following theorem.

Theorem 7.3. *Let* (X,c) *be a* G_0*-closure space. Let* X^* *be a collection of grills on* X *satisfying* $X^* \supset [\mathcal{G}_c(x): x \in X]$. *Then* $(\varphi,(X^*,h_r))$ *is a* G_0*-extension of* (X,c) *with trace system* X^*. *Moreover, all extensions on* X^* *with trace system* X^* *can be obtained by suitable choice of* r.

8. Principal Extensions

If (X^*,h_r) is to be a topological space then all $h_r(\alpha)$ must be closed sets. Hence, in particular, all A^*, $A \subset X$, must be closed. We have

$$(A \cup B)^* = A^* \cup B^*$$

so that $[A^*: A \subset X]$ can be taken as a base for the closed sets of a topology on X^*. The corresponding closure operator

$$g(\alpha) = \cap[A^*: A^* \supset \alpha], \ \alpha \subset X^*,$$

is the largest closure operator on X^* for which the extension $(\varphi,(X^*,g))$ has the trace system X^*. This extremal property justifies giving the extension $(\varphi,(X^*,g))$ a special name. It is called the *principal (strict) extension of* (X,c) *with trace system* X^*.

For closure spaces the above approach does not work for a number of reasons:

(i) $h_r(\alpha)$, $\alpha \subset X^*$ need not be closed sets.

(ii) Closure spaces are not determined by their closed sets and thus have no bases.

(iii) g always defines a Kuratowski closure operator, since it is a hull operator. Thus $(\varphi,(X^*,g))$ cannot even be an extension of (X,c) if c is not a Kuratowski closure operator.

Nevertheless (and rather surprisingly) a meaningful principal extension with respect to a given trace system X^* can be defined for closure spaces by choosing r in h_r to be

$$r(\beta) = \cap[A^*: A^* \supset \beta], \ \beta \subset X^* \sim \varphi(X).$$

One then has

$$h_r(\beta) = (\varphi^{-1}(\alpha))^* \cup \cap[A^*: A^* \supset \alpha], \ \alpha \subset X^*.$$

Since this reduces to g in case (X^*,h_r) is topological, we define in general

$$g(\alpha) = (\varphi^{-1}(\alpha))^* \cup \cap[A^*: A^* \supset \alpha],$$

and say $(\varphi,(X^*,g(\alpha)))$ is the *principal extension* for all closure spaces (X,c) and given trace system X^*.

It is shown in CNT that even in the general case principal extensions have extremal properties. They are:

I) $\beta \subset X^* \sim \varphi(X)$, $A \subset X$, $\beta \subset g(\varphi(A)) \Rightarrow g(\beta) \subset g(\varphi(A))$.

II) For every $\beta \subset X^* \sim \varphi(X)$ there is a family $\mathcal{A}_\beta \subset \mathcal{P}(X)$ such that

$$g(\beta) = \cap[g(\varphi(A)): A \in \mathcal{A}_\beta].$$

g is the largest closure operator satisfying I and the smallest satisfying II.

As was shown by Reed [1978] in the case of topological spaces, principal extensions play a significant role in establishing a mapping from near structures to various kinds of compactifications. In Section 10 we shall report on work of CNT showing that principal extensions on closure spaces can be used for a similar purpose.

We conclude this section by observing that in the case of principal extensions Theorem 7.2 can be strengthened to the following:

Let $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B}_{\mathcal{A}} = [\beta: \beta \in X^*, \varphi^{-1}(\beta) \in \mathcal{A}]$ then (X^*,g) is a $\mathcal{B}_{\mathcal{A}}/\mathcal{P}(\varphi(X))$ space provided

$$D^{\mathcal{A}}(\mathcal{G}) \subset \mathcal{G}, \forall \mathcal{G} \in X^*.$$

9. Near Structures

A family $\nu \subset \mathcal{P}(\mathcal{P}(X))$ is called a *near structure* (or *nearness*) *on* X if

N_1: $\mathcal{A} \subset \mathcal{P}(X)$, $\cap[A: A \in \mathcal{A}] \neq \phi \Rightarrow \mathcal{A} \in \nu$.

N_2: $\mathcal{B} \subset \mathcal{P}(X)$, $\mathcal{B} \in \nu \Rightarrow [A: A \subset X, \exists B \in \mathcal{B}, A \supset B] \in \nu$.

N_3: $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{B} \subset \mathcal{P}(X)$, $[A \cup B: A \in \mathcal{A}, B \in \mathcal{B}] \in \nu \Rightarrow \mathcal{A} \in \nu$ or $\mathcal{B} \in \nu$.

N_4: $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{A} \in \nu \Rightarrow \phi \notin \mathcal{A}$.

Near structures are generalizations of proximity and contiguity structures. In near structures "nearness" is defined for sets of arbitrary cardinality, while in contiguities it is defined only for finite collections of sets and in proximities only for pairs of sets.

A near structure ν induces a closure operator

$$c_\nu(A) = [x: \{[x],A\} \in \nu].$$

(X,c_ν) is a closure space. Until recently it was considered desirable to add to the axioms for a near structure the following

N_5: $\mathcal{A} \subset \mathcal{P}(X)$, $[c_\nu(A): A \in \mathcal{A}] \in \nu \Rightarrow \mathcal{A} \in \nu$.

We call such nearnesses LO-near structures. (Herrlich [1974] made N_5 part of his definition of a near structure.) If ν is a LO-nearness then (X,c_ν) is a topological space. However, for many purposes a slightly weaker additional condition

$$N_6\colon \mathcal{A} \subset \mathcal{P}(X),\ \cap[c_\nu(A)\colon A \in \mathcal{A}] \neq \phi \Rightarrow \mathcal{A} \in \nu$$

suffices. Near structures satisfying N_1, N_2, N_3, N_4 and N_6 are called RI-*near structures*. In this case (X,c_ν) need not be a topological space. It is, however, an S-space as well as an H_2-space.

If $\mathcal{G} \in \Gamma(X)$ and $\mathcal{G} \in \nu$ we call it a ν-*clan*. Every maximal family in ν will be called a ν-*cluster*. It is known that every maximal family in ν is a grill. Thus every ν-cluster is a ν-clan. Not every $\mathcal{A} \in \nu$ need to be contained in a maximal family. Near structures ν for which every $\mathcal{A} \in \nu$ is contained in a ν-cluster are called *cluster generated*.

The importance of RI-near structures derives from the following characterization.

Theorem 9.1. *A nearness* ν *on* X *is a* RI-*nearness iff for all* $x \in X$ *the adherence grills* $\mathcal{G}_{c_\nu}(x)$ *are* ν-*clusters*.

It will be convenient to set

$$X^\nu = [\mathcal{G}\colon \mathcal{G} \text{ is a } \nu\text{-cluster}].$$

We also shall call a grill $\mathcal{G} \in \Gamma(X)$ a ξ_ν-*clan* if every finite subfamily of $\mathcal{G}$ belongs to ν. A π_ν-*clan* shall be a grill $\mathcal{G}$ such that every two element subfamily of $\mathcal{G}$ belongs to ν. In terms of these concepts we now define:

ν is called a *proximal nearness* if

$$\mathcal{A} \in \nu \text{ iff } \mathcal{A} \subset \mathcal{G}, \text{ where } \mathcal{G} \text{ is a } \pi_\nu\text{-clan.}$$

ν is called a *contigual nearness* if

$$\mathcal{A} \in \nu \text{ iff } \mathcal{A} \subset \mathcal{G}, \text{ where } \mathcal{G} \text{ is a } \xi_\nu\text{-clan.}$$

ν is called *weakly contigual* if

$[c_\nu(A): A \in \mathcal{A}] \in \nu$ if $\mathcal{A} \subset \mathcal{G}$, where $\mathcal{G}$ is a ξ_ν-clan and $\mathcal{U} \in \Omega(X) \Rightarrow \mathcal{U} \in \nu$.

In the next section we shall show how these kinds of near structures correspond to different kinds of compactifications.

10. A Correspondence

Reed [1978] studied a correspondence between LO-near structures and principal extensions on T_1-spaces. Here we give a brief account of the extension of this result to RI-near structures on D_1-closure spaces obtained by CNT.

The mapping in question

$$E_\nu = (\varphi, (X^\nu, g))$$

has been investigated by Bentley, Herrlich, Naimpally and probably others. CNT raised the question how general the underlying space (X,c) and the nearness ν, with $c_\nu = c$, could be and still have the mapping defined and 1-1. Clearly (X,c) must be a G_0-space. Since X^ν is to be the trace-system of the extension one must have

$$X^\nu \supset [\mathcal{G}_{c_\nu}(x): x \in X].$$

Thus ν must be a RI-nearness. Then (X,c) is a $G_0 \cap S \cap H_2$-space, that is according to Theorem 5.3, a D_1-space. Finally, in order for E_ν to be 1-1 we want ν to be cluster generated. For every D_1-closure space (X,c) the mapping E_ν defined on all cluster generated RI-near structures ν, with $c_\nu = c$, is 1-1 and onto all principal D_1-extensions of the space (X,c).

So far there is nothing special in the choice of principal extensions. Any other extension, so long as it is completely determined by its trace system could also have been selected. However beyond wanting to have a 1-1 correspondence, the aim is to identify the near structures whose images are certain compactifications. There is then an immediate difference here, since, in the topological case, one needs to distinguish only two kinds of compactness while there are three kinds for closure spaces. Secondly, principal extensions appear not to be very significant in case of closure spaces and so it was questionable whether they would be suitable in the more general case. That they indeed can be used is probably the most unexpected result of CNT. The theorem obtained is as follows

Theorem 10.1. *Let* (X,c) *be a fixed* D_1*-space. The function* E_ν *maps the*

$$\left\{\begin{array}{l} \textit{proximal} \\ \textit{contigual} \\ \textit{weakly contigual and cluster generated} \end{array}\right\} \text{RI-}\textit{near structures}$$

$$\textit{on } (X,c) \text{ 1-1 } \textit{onto the} \left\{\begin{array}{c} \textit{linkage compact} \\ \text{F-}\textit{linkage compact} \\ \textit{compact} \end{array}\right\} \textit{principal } D_1\textit{-extensions}$$

of (X,c).

11. More About Convergence of Grills

In defining convergence of grills as is done in Section 2 the author does not intend to break new ground in convergence theory. We mean to define convergence *only* for grills, just as in classical convergence theory it is defined *only* for filters.

In closure spaces "closure" and hence "adherence grills" appear as the primary concepts and it thus seems desirable to develop a convergence theory for closure spaces in terms of grills rather than in terms of filters. This also is the reason why we presented Kent's results on convergence structures (in Section 3) in terms of grills.

In the context of nearness spaces Herrlich (8) found it convenient to define "convergence" and "cluster points" for arbitrary families of sets. In a somewhat related setting convergence structures and grills have recently been studied by the following:

W. A. Robertson, *Convergence as a nearness concept*, Thesis, Carlton University, 1975.

H. L. Bentley, H. Herrlich, and W. A. Robertson, *Convenient categories for topologists*, Comment. Math. Univ. Carolinae 17 (1976), 207-227.

F. Schwarz, *Connections between convergence and nearness*, Categorical Topology, Proceedings of the International Conference, Berlin 1978, Lecture Notes in Mathematics 719 (1979), 345-357.

12. Further Results

Two articles of Chattopadhyay and Njåstad, just completed, further support our thesis that to require, that the underlying spaces be topological, is not natural.

In the first of these papers they study extensions of near maps. Earlier work had been in terms of LO-near structures, theirs is in terms of RI-nearnesses.

In the second paper Chattopadhyay and Njåstad investigate contraction maps and fixed points in a very general setting.

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