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Research Announcement:

REPORT ON SPANS

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REPORT ON SPANS

A. Lelek

Almost 20 years ago a study of a numerical concept, called the span, for metric spaces was originated [12], and since that time a considerable progress has been made regarding the concept itself and its interaction with other notions of metric topology. The purpose of this paper[1] is to review the existing literature on spans and to announce some recent results in the theory of spans of continua.

I. Span and Semispan

The span can be interpreted as a connectedness-type analogue of the diameter of an object. Suppose X is a metric space. We define the *span* $\sigma(X)$ of X to be the least upper bound of the set of real numbers α such that there exist continuous mappings $f,g: C \to X$, where C is connected, $f(C) = g(C)$ and $\mathrm{dist}[f(c),g(c)] \geq \alpha$ for $c \in C$. The *semispan* $\sigma_0(X)$ of X is defined similarly with the equality $f(C) = g(C)$ relaxed to the inclusion $f(C) \subset g(C)$. Two other relevant notions, the surjective span $\sigma^*(X)$ and the surjective semi-span $\sigma_0^*(X)$, have also been identified [16]; they arise by imposing an additional requirement of $g(C) = X$ in both definitions, respectively. The surjective span may differ from the span even for some rather primitive objects [14,16].

[1]The paper is a slightly expanded version of the invited talk given by the author during the Fifteenth Spring Topology Conference at Virginia Polytechnic Institute and State University, on March 19, 1981.

Obviously, $\sigma(X) \leq \sigma_0(X)$. There exists a simple 4-od X in the Euclidean 3-space such that $\sigma(X) = \frac{1}{2}$ and $\sigma_0(X) = 1$ (see [16], p. 37). A continuous mapping f of a continuum X onto a continuum Y is said to be *weakly confluent* provided each subcontinuum of Y is the image under f of a subcontinuum of X. We say that a continuum Y is in Class (W) provided each continuous mapping of any continuum onto Y is weakly confluent.

Theorem (J. F. Davis). *If* X *is a continuum such that every subcontinuum of* X *is in* Class (W), *then* $\sigma(X) = \sigma_0(X)$. (See [1].)

Corollary. For each continuum X, $\sigma(X) = 0$ *if and only if* $\sigma_0(X) = 0$.

A related quantity, called the symmetric span, is studied by J. F. Davis [1,3]. Also, the span theory can be somewhat generalized to the effect that it facilitates the study of certain geometric phenomena involving homotopically essential mappings and expansions of continua into inverse sequences of polyhedra. For that purpose, the notion of the span of a continuous mapping [5,18] has been found useful.

II. Union Theorems

It is not difficult to show that all chainable continua (i.e., those which admit ε-mappings into the real line for each $\varepsilon > 0$) have span zero. Actually, there is a correlation between the spans of the domain and range spaces and the distances of some points in them (see [12], p. 209; compare [16], p. 39). It is unknown, however, whether or not all continua of span zero are chainable [13]; we know only that they are tree-like [18].

Suppose X and Y are chainable continua which intersect in a non-empty continuum. A well-known result states that then the union $X \cup Y$ is chainable provided it is atriodic[2]. The next theorem establishes a similar fact for continua of semispan zero.

Theorem (E. Duda and J. Kell). *If* X *and* Y *are continua such that* $\sigma_0(X) = \sigma_0(Y) = 0$, $X \cap Y$ *is a non-empty continuum and* $X \cup Y$ *is atriodic, then* $\sigma_0(X \cup Y) = 0$. (See [4].)

By the corollary mentioned in Section I, the semispan σ_0 in this theorem can be replaced by the span σ. For continuous mappings of some continua onto continua with span zero or semispan zero, the sets of distances between points having the same images can be determined using only the span or the semispan of the domain space, respectively [15, 16,17,19]. For continua on the plane, a special technique has been developed by L. G. Oversteegen and E. D. Tymchatyn [21] leading to objects with small spans. It is used [21] to show, among other things, that all non-separating planar homogeneous continua have span zero. In a different manner, J. F. Davis [2] proved that each almost chainable homogeneous continuum has span zero. Afterwards, however, W. Lewis [20]

[2]That theorem was proved by W. T. Ingram, Fund. Math. 63 (1968), 193-198; see also J. B. Fugate, Trans. Amer. Math. Soc. 123 (1966), 460-468, where an earlier form of it was discovered.

proved that there is only one such a continuum, namely, the pseudo-arc.

III. Weak Chainability and Span Zero

A continuum is called *weakly chainable* provided it is a continuous image of a chainable continuum or, equivalently, of the pseudo-arc. It is an open problem (raised by L. K. Mohler) whether or not all weakly chainable, atriodic, tree-like continua are chainable. Testing, as in Section II, the idea of close affinity between chainable continua and continua of span zero, one could ask a corresponding question to which the following theorem gives a partial answer.

Theorem (L. G. Oversteegen and E. D. Tymchatyn). *If* X *is a weakly chainable, atriodic, tree-like continuum such that every proper subcontinuum of* X *is chainable, then* $\sigma(X) = 0$. (See [22].)

Many interesting examples of atriodic tree-like continua with positive span and with chainable proper subcontinua have been constructed and investigated by W. T. Ingram [5,6,7,8,9,10]. By the last theorem, none of them is weakly chainable. On the other hand, he [11] has also found tree-like continua whose non-degenerate subcontinua all have positive span.

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