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Research Announcement: METRIC SPACES WITH INTRINSIC GEOMETRICAL STRUCTURE

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Research Announcement

METRIC SPACES WITH INTRINSIC GEOMETRICAL STRUCTURE

Juliusz Oledzki

The following concepts were introduced by K. Borsuk and studied by S. Spiez and J. Oledzki (to appear in Bull. Polon. Acad. Sci.).

Given an arc A with a parametric representation $p: [0,1] \to A$ in a metric space (X,ρ). The length of A, $|A|$, is the least upper bound of $\sum_{i=0}^{k} \rho(p(t_i), p(t_{i+1}))$ where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$.

A space X is said to be geometrically acceptable ($X \in GA$) if every pair of points $x, y \in X$ can be joined by an arc with finite length and for every point $x \in X$ and every number $\varepsilon > 0$ there is a neighborhood U of x in X such that every point $y \in U$ and the point x can be joined by an arc of length $< \varepsilon$. Any space $X \in GA$ can be metrized by the intrinsic metric $\rho_X(x,y) = \inf\{|A|; A$ is an arc in X joining points x and y$\}$.

If (X,ρ) and (Y,ρ') are geometrically acceptable spaces with intrinsic metrics ρ_X and ρ_Y respectively, then the intrinsic metric $\rho_{X\times Y}$ of the space $(X \times Y, \rho'')$ is given by $\rho_{X\times Y}((x_1,y_1),(x_2,y_2)) = \sqrt{(\rho_X(x_1,x_2))^2 + (\rho_Y(y_1,y_2))^2}$, where $\rho''((x_1,y_1),(x_2,y_2)) = \sqrt{(\rho(x_1,x_2))^2 + (\rho'(y_1,y_2))^2}$.

A map f from a space $X \in GA$ onto a space $Y \in GA$ is said to be an intrinsic isometry if $\rho_X(x,y) = \rho_Y(f(x),f(y))$, for every $x, y \in X$.

It is known that if a Riemannian surface (with a metric at least two times differentiable) is intrinsically isometric to the 2-dimensional sphere then it is isometric to the sphere. Without assumption about differentiability there is a new possibility to construct intrinsic isometries.

Example 1. Suppose X is a subspace of Euclidean space E^n, the map $s: E^n \to E^n$ is a symmetry with respect to an n-1-dimensional hyperplane H and let $X \cap s(X) \subset H$. The formula

$$f(x) = \begin{cases} x & \text{if } x \in H^+ \\ s(x) & \text{if } x \notin H^+ \end{cases}$$

where H^+ is a half-space with the boundary H, describes the intrinsic isometry called the reflection.

Example 2. The Euclidean space E^1 is intrinsically isometric to a subset L of E^2 with an arbitrarily small diameter. To construct the intrinsic isometry, divide E^1 into small segments and map isometrically each segment onto appropriate segment of the arc L pictured below (Fig. 1).

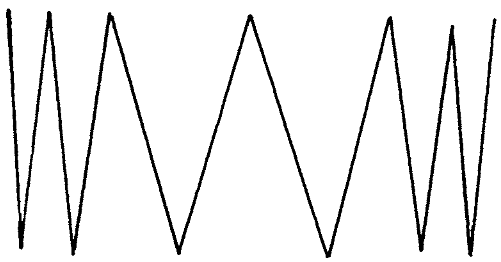

Fig. 1

The arc L can also be obtained from a straight line by reflections with respect to lines perpendicular to the bisectrices of the angles between neighbor segments of the arc L. By induction with respect to n we can prove the following.

Theorem 1. *The Euclidean space* E^n *is intrinsically isometric to a subspace of* E^{n+1} *with an arbitrarily small diameter* (see Fig. 2).

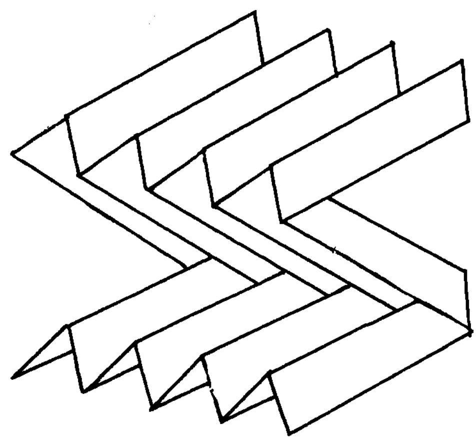

Fig. 2

Theorem 2. *Every* 1-*dimensional (countable) polyhedron is intrinsically isometric to a subspace of* E^3 *with an arbitrarily small diameter.*

Proof in case of finite polyhedron. A 1-dimensional finite polyhedron can be embedded in E^3 by a simplicial map decreasing the distances between any vertices. Then we can replace each edge by an arc with primary length as in Fig. 3.

Problem. Is every n-dimensional, geometrically acceptable, separable, metric space intrinsically isometric to a subspace of E^{2n+1}? (Or to a subspace with arbitrarily small diameter?)

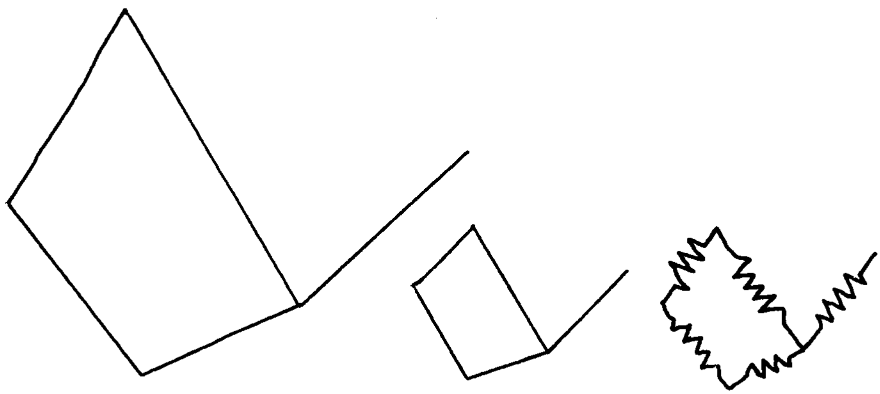

Fig. 3

Remark. If an n-dimensional polyhedron P is already contained in E^{2n+1} then P is intrinsically isometric to an arbitrarily small subset. We can decrease its diameter by the composition of several reflections.

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