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AANR'S and ARI Maps

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1. Introduction

We use results of Čerin to examine shape properties of refinable and approximately right invertible maps. Relations between certain function spaces and hyperspaces are also examined.

2. Preliminaries

A compactum X is called an approximate absolute neighborhood retract in the sense of Noguchi (AANR_N) if whenever X is embedded in an ANR M , there is a neighborhood U of X in M such that for every $\epsilon > 0$, there is an ϵ -retraction of U to X , i.e., a continuous function $f: U \rightarrow X$ such that $f|_X$ is an ϵ -push (moves no point by more than ϵ). If it may always be assumed that $f|_X$ is a surjection, then X is called a surjective approximate absolute neighborhood retract in the sense of Noguchi (SAANR_N).

If in the above, U may be taken to be M , we say X is an approximate absolute retract (AAR) or a surjective approximate absolute retract (SAAR), respectively.

If for every $\epsilon > 0$ there exists a neighborhood U of X in M such that there is an ϵ -retraction $g: U \rightarrow X$, then X is called an approximate absolute neighborhood retract in the sense of Clapp (AANR_C). If it may always be assumed that $g|_X$ is a surjection, then X is called a surjective approximate absolute neighborhood retract in the sense of Clapp (SAANR_C).

The definitions above are from [N], [C1], and [P1].

A continuous surjection $r: X \rightarrow Y$ is called refinable [F-R] if for every $\varepsilon > 0$, there is an ε -map $f: X \rightarrow Y$ (i.e., $\text{diam } f^{-1}(y) < \varepsilon$ for all $y \in Y$) that is ε -close to r .

A map $f: X \rightarrow Y$ between compacta is approximately right invertible (ARI) [G] if for every $\varepsilon > 0$ there is a map $g_\varepsilon: Y \rightarrow X$ such that fg_ε is ε -close to l_Y . If also there exists $G_\varepsilon: X \rightarrow Y$ such that $g_\varepsilon G_\varepsilon$ is ε -close to l_X , then f is approximately invertible (AI) [Ce3].

A compactum X is calm [Ce1] if whenever $X \subset M \in \text{ANR}$, there is a neighborhood V of X in M such that for every neighborhood U of X in M there is a neighborhood W of X in M , $W \subset U$, such that if $f, g: Y \rightarrow U$ with $f \approx g$ in V , then $f \approx g$ in U for all $Y \in \text{ANR}$. We have:

(2.1) *Theorem [Ce-S]. A compactum X is an FANR if and only if X is calm and movable.*

Let 2^Y be the set of all nonempty compact subsets of a metric space Y . The metric of continuity d_C is defined in [Bk1] as follows: $d_C(A, B) = \varepsilon$ if ε is the infimum of those nonnegative t such that there are t -pushes $f: A \rightarrow B$ and $g: B \rightarrow A$.

If $A, B \in 2^Y$ and there are continuous surjections $f: A \rightarrow B$ and $g: B \rightarrow A$, then the metric of continuous surjection $d_{\overline{C}}$ is defined (see [Ce2]) by: $d_{\overline{C}}(A, B) = \varepsilon$ if ε is the infimum of those nonnegative t such that there are surjective t -pushes $f: A \rightarrow B$ and $g: B \rightarrow A$.

The Hilbert cube is denoted by Q .

3. Shape Properties of Certain Maps

Let X and Y be compacta in AR-spaces M and N , respectively. Let us recall the definitions of *quasi-domination* and *quasi-equivalence* [Bk2]: X quasi-dominates Y ($X \underset{-q}{>} Y$) if for every neighborhood U of Y in N there exists a neighborhood V of Y in N , $V \subset U$, and fundamental sequences $\underline{f} = \{f_k, X, Y\}_{M, N}$, $\underline{g} = \{g_k, Y, X\}_{N, M}$ such that for almost all k , $f_k g_k|_V \simeq i_{V, U}$ in U , where $i_{V, U}$ is the inclusion of V into U .

If for every neighborhood (W, U) of (X, Y) in (M, N) there exist neighborhoods (W_1, V) of (X, Y) in (M, N) , $W_1 \subset W$, $V \subset U$, and fundamental sequences \underline{f} , \underline{g} as above with $f_k g_k|_V \simeq i_{V, U}$ in U and $g_k f_k|_{W_1} \simeq i_{W_1, W}$ in W for almost all k , then X and Y are quasi-equivalent ($X \underset{q}{\simeq} Y$). These notions are in general weaker than shape domination and shape equivalence, respectively, but they coincide when Y is calm (when X and Y are calm, respectively) [Bx2].

(3.1) *Theorem.* Let X and Y be compacta and let $f: X \rightarrow Y$ be ARI. Then $X \underset{-q}{>} Y$. If f is AI, then $X \underset{q}{\simeq} Y$.

Proof. There is no loss of generality in assuming $M = N = Q$ [Bk2]. Let U be a neighborhood of Y in Q . There is a compact ANR neighborhood V' of Y in Q , $V' \subset U$, and an $\epsilon > 0$ such that ϵ -close maps into V' are homotopic. Since f is ARI, there exist $g_\epsilon: Y \rightarrow X$ such that fg_ϵ is an ϵ -push. Let \underline{f} , \underline{g} be fundamental sequences generated by f and g_ϵ , respectively. There is a neighborhood V of Y in Q , $V \subset V'$, such that $f_k g_k|_V$ is an ϵ -push and $f_k g_k(V) \subset V'$ for almost all k . By choice of ϵ , $f_k g_k|_V \simeq i_{V, U}$ in V' ,

hence in U , for almost all k . Thus $X \underset{Q}{\geq} Y$.

If f is AI, then for a neighborhood (W,U) of (X,Y) in (Q,Q) there exist compact ANR neighborhoods W', V' of X and Y in Q , respectively, $W' \subset W$, $V' \subset U$, and an $\varepsilon > 0$ such that ε -close maps into either of W' or V' are homotopic. Since f is AI, there exist $g_\varepsilon: Y \rightarrow X$ and $f_\varepsilon: X \rightarrow Y$ such that fg_ε is $(\varepsilon/2)$ -close to 1_Y , $g_\varepsilon f_\varepsilon$ is δ -close to 1_X , where $0 < \delta < \varepsilon$ and $d(x_1, x_2) < \delta$ implies $d(f(x_1), f(x_2)) < \varepsilon/2$. Therefore, for $x \in X$,

$$\begin{aligned} d(f(x), f_\varepsilon(x)) &\leq d(f(x), fg_\varepsilon f_\varepsilon(x)) + d(fg_\varepsilon f_\varepsilon(x), \\ & f_\varepsilon(x)) < (\text{by choice of } \delta) \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus f and f_ε are ε -close.

If \underline{f} , \underline{g} , and \underline{F} are fundamental sequences generated by f , g_ε , and f_ε , respectively, it follows from our choice of ε that there is a neighborhood (W_1, V) of (X, Y) in (Q, Q) , $W_1 \subset W'$, $V \subset V'$, such that $F_k g_k|_{W_1} \approx f_k g_k|_{W_1} \approx i_{W_1, W}$ in W' , hence in W , and (by our choice of δ) such that $g_k F_k|_V \approx i_{V, U}$ in V' , hence in U .

It follows that $X \underset{Q}{\approx} Y$.

(3.2) *Corollary.* Let X and Y be compacta, $f: X \rightarrow Y$ an ARI map. Then

- a) if Y is calm, $\text{Sh } X \underset{\geq}{\geq} \text{Sh } Y$.
- b) if f is AI and X is calm, $\text{Sh } X \underset{\leq}{\leq} \text{Sh } Y$.
- c) if f is AI and X and Y are both calm, $\text{Sh } X = \text{Sh } Y$.

Proof. This follows immediately from (3.1) and [Bx2, (3.3)].

The following is suggested by [F-R, 3.4] and gives a partial answer to [F-R, Question 5]:

(3.3) *Corollary.* Let $X \in \text{SAANR}_C$ and let $r: X \rightarrow Y$ be a refinable map.

- a) If $X \in \text{SAANR}_N$, $\text{Sh } Y \geq \text{Sh } X$.
- b) If $Y \in \text{SAANR}_N$, $\text{Sh } X \geq \text{Sh } Y$.
- c) If $X, Y \in \text{SAANR}_N$, $\text{Sh } X = \text{Sh } Y$.

Proof. We have $X \in \text{SAANR}_N$ if and only if $X \in \text{SAANR}_C$ and $X \in \text{FANR}$ [Ce2], $X \in \text{AANR}_C$ implies X is movable [Bg, Theorem 6], and (2.1) imply: $X \in \text{SAANR}_N$ if and only if $X \in \text{SAANR}_C$ and X is calm.

It follows from [Ce3, opening remarks in §5] that f is AI. The assertions follow from (3.2).

We remark that it follows that Y in (3.3) is an SAANR_C , by [P2, Theorem 2].

For the collection of ARI maps that are strongly approximately right invertible (SARI) and for the collection of AI maps that are strongly approximately invertible (SAI) (see [Ce3] for definitions) we have:

(3.4) *Corollary.* Let $f: X \rightarrow Y$ be a map between compacta.

- a) If $X \in \text{AANR}_N$ and f is SARI, $\text{Sh } X \geq \text{Sh } Y$.
- b) If one of X or Y is an AANR_N and f is SAI, $\text{Sh } X = \text{Sh } Y$.

Proof. We use the fact that $X \in \text{AANR}_N$ implies $X \in \text{FANR}$ [Gm], and therefore by (2.1) X is calm.

a) We have $Y \in \text{AANR}_N$ [Ce3, (5.2a)], hence Y is calm. The assertion follows from (3.2a).

b) We have both $X, Y \in \text{AANR}_N$ [Ce3, (5.2b)]. The assertion follows from (3.2c).

4. Some Function Spaces and Hyperspaces

In this section we assume X is a compactum, Y a metric space, and Y^X is the space of maps from X to Y with the compact-open (=sup-metric) topology. Suppose $\{f_i\}_{i=0}^\infty \subset Y^X$ with $A_i = f_i(X)$ for all i . What does $\lim_{i \rightarrow \infty} f_i = f_0$ imply about $\{A_i\}_{i=0}^\infty$ with respect to hyperspaces? It is clear that $\lim_{i \rightarrow \infty} A_i = A_0$ in the topology of the Hausdorff metric. Borsuk's example [B1] of arcs converging to S^1 in the Hausdorff metric but not d_C may be used to construct a convergent sequence $f_i \rightarrow f_0$ in $(S^1)^I$ such that $A_0 \neq \lim_{i \rightarrow \infty} A_i$ in d_C . The approach of [Bx1], that by restricting X or by considering appropriate subspaces of Y^X we may obtain interesting results, is used here.

Let us define $R(X,Y)$ and $ARI(X,Y)$ to be the subspaces of Y^X consisting of those maps f such that $f: X \rightarrow f(X)$ is refinable or ARI, respectively. Let d^S be the sup-metric for Y^X .

(4.1) *Theorem.* Let $X, Y, \{f_i\}_{i=0}^\infty, \{A_i\}_{i=0}^\infty$ be as above. Suppose $X \in \text{SAANR}_C$ and $f_i \in R(X,Y)$ for $i > 0$. If $f_0 \in R(X,Y)$ then $\lim_{i \rightarrow \infty} d_C^-(A_i, A_0) = 0$.

Proof. Suppose $f_0 \in R(X,Y)$. Let $\varepsilon > 0$. Since $X \in \text{SAANR}_C$, there are [P2, Theorem 1] continuous surjections $g_i: A_i \rightarrow X$ such that $f_i g_i: A_i \rightarrow A_i$ is an $(\varepsilon/2)$ -push for all i .

Fix i such that $d^S(f_i, f_0) < \varepsilon/2$.

Consider the continuous surjections $F_i = f_0 g_i: A_i \rightarrow A_0$, $G_i = f_i g_0: A_0 \rightarrow A_i$. We have $d^S(F_i, l_{A_i}) \leq d^S(f_0 g_i, f_i g_i) + d^S(f_i g_i, l_{A_i}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, and $d^S(G_i, l_{A_0}) \leq$

$$d^S(f_i g_o, f_o g_o) + d^S(f_o g_o, l_{A_o}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that $\lim_{i \rightarrow \infty} d_C(A_i, A_o) = 0$.

(4.2) *Theorem.* Suppose $\lim_{i \rightarrow \infty} f_i = f_o$ in Y^X with $f_i \in \text{ARI}(X, Y)$ for $i > 0$. Then $f_o \in \text{ARI}(X, Y)$ if and only if $\lim_{i \rightarrow \infty} d_C(A_i, A_o) = 0$.

Proof. Suppose $f_o \in \text{ARI}(X, Y)$. Let $\epsilon > 0$. Fix i such that $d^S(f_i, f_o) < \epsilon/2$. There exist maps $h_i: A_i \rightarrow X$, $h_o: A_o \rightarrow X$ such that $f_i h_i$ and $f_o h_o$ are $(\epsilon/2)$ -pushes. Consider $F = f_o h_i: A_i \rightarrow A_o$, $G = f_i h_o: A_o \rightarrow A_i$.

We have

$$d^S(F, l_{A_i}) \leq d^S(f_o h_i, f_i h_i) + d^S(f_i h_i, l_{A_i}) < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{and}$$

$$d^S(G, l_{A_o}) \leq d^S(f_i h_o, f_o h_o) + d^S(f_o h_o, l_{A_o}) < \epsilon/2 + \epsilon/2 = \epsilon.$$

It follows that $\lim_{i \rightarrow \infty} d_C(A_i, A_o) = 0$.

Conversely, if $\lim_{i \rightarrow \infty} d_C(A_i, A_o) = 0$, fix $\epsilon > 0$ and i such that $d_C(A_i, A_o) < \epsilon/3$ and $d^S(f_i, f_o) < \epsilon/3$. There exist maps $g_i: A_o \rightarrow A_i$, $h_i: A_i \rightarrow X$ such that g_i and $f_i h_i$ are $(\epsilon/3)$ -pushes. Consider $h = h_i g_i: A_o \rightarrow X$. We have $d^S(f_o h, l_{A_o}) \leq d^S(f_o h_i g_i, f_i h_i g_i) + d^S(f_i h_i g_i, g_i) + d^S(g_i, l_{A_o}) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

It follows that $f_o \in \text{ARI}(X, Y)$.

References

[Bg] S. A. Bogatyĭ, *Approximate and fundamental retracts*, Mat. Sbornik 93 (135) (1974), 91-103.
 [Bkl] K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fund. Math. 41 (1954), 168-202.

- [Bk2] _____, *Some quantitative properties of shapes*, *Fund. Math.* 93 (1976), 197-212.
- [Bx1] L. Boxer, *Retraction spaces and the homotopy metric*, *Topology and Its Appl.* 11 (1980), 17-29.
- [Bx2] _____, *Remarks on quasi-domination* (preprint).
- [Cel] Z. Čerin, *Homotopy properties of locally compact spaces at infinity--calmness and smoothness*, *Pacific J. of Math.* 79 (1978), 69-91.
- [Ce2] _____, *Surjective approximate absolute (neighborhood) retracts*, *Top. Proc.* 6 (1981), 5-24.
- [Ce3] _____, C_p -*e-movable and C -e-calm compacta and their images*, *Compositio Math.* (to appear).
- [Ce-S] _____ and A. P. Šostak, *Some remarks on Borsuk's fundamental metric*, in *Topology*, v. 1 (Á. Czászár, ed.), 233-252, North-Holland, New York, 1980.
- [Cl] M. Clapp, *On a generalization of absolute neighborhood retracts*, *Fund. Math.* 70 (1971), 117-130.
- [F-R] J. Ford and J. W. Rogers, Jr., *Refinable maps*, *Colloquium Math.* 39 (1978), 263-269.
- [G] R. Geoghegan, ed., *Open problems in infinite-dimensional topology*, *Top. Proc.* 4 (1979), 287-338.
- [Gm] A. Gmurczyk, *Approximative retracts and fundamental retracts*, *Colloquium Math.* 23 (1971), 61-63.
- [N] H. Noguchi, *A generalization of absolute neighborhood retracts*, *Kodai Math. Sem. Reports* 1 (1953), 20-22.
- [Pl] P. Patten, *Locally connected generalized absolute neighborhood retracts*, *Top. Proc.* 3 (1978), 159-168.
- [P2] _____, *Refinable maps and generalized absolute neighborhood retracts* (preprint).

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