TOPOLOGY PROCEEDINGS
Volume 6, 1981
Pages 219–226
http://topology.auburn.edu/tp/

AANR'S and ARI Maps

by
Laurence Boxer

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

AANR'S and ARI Maps

Laurence Boxer

1. Introduction

We use results of Čerin to examine shape properties of refinable and approximately right invertible maps. Relations between certain function spaces and hyperspaces are also examined.

2. Preliminaries

A compactum X is called an approximate absolute neighborhood retract in the sense of Noguchi ($AANR_N$) if whenever X is embedded in an ANR M, there is a neighborhood U of X in M such that for every $\varepsilon > 0$, there is an ε-retraction of U to X, i.e., a continuous function $f: U \to X$ such that $f|X$ is an ε-push (moves no point by more than ε). If it may always be assumed that $f|X$ is a surjection, then X is called a surjective approximate absolute neighborhood retract in the sense of Noguchi ($SAANR_N$).

If in the above, U may be taken to be M, we say X is an approximate absolute retract (AAR) or a surjective approximate absolute retract (SAAR), respectively.

If for every $\varepsilon > 0$ there exists a neighborhood U of X in M such that there is an ε-retraction $g: U \to X$, then X is called an approximate absolute neighborhood retract in the sense of Clapp ($AANR_C$). If it may always be assumed that $g|X$ is a surjection, then X is called a surjective approximate absolute neighborhood retract in the sense of Clapp ($SAANR_C$).

The definitions above are from [N], [Cl], and [Pl].

A continuous surjection $r: X \to Y$ is called refinable [F-R] if for every $\varepsilon > 0$, there is an ε-map $f: X \to Y$ (i.e., diam $f^{-1}(y) < \varepsilon$ for all $y \in Y$) that is ε-close to r.

A map $f: X \to Y$ between compacta is approximately right invertible (ARI) [G] if for every $\varepsilon > 0$ there is a map $g_\varepsilon: Y \to X$ such that fg_ε is ε-close to 1_Y. If also there exists $G_\varepsilon: X \to Y$ such that $g_\varepsilon G_\varepsilon$ is ε-close to 1_X, then f is approximately invertible (AI) [Ce3].

A compactum X is calm [Ce1] if whenever $X \subset M \in$ ANR, there is a neighborhood V of X in M such that for every neighborhood U of X in M there is a neighborhood W of X in M, $W \subset U$, such that if $f,g: Y \to U$ with $f \simeq g$ in V, then $f \simeq g$ in U for all $Y \in$ ANR. We have:

(2.1) *Theorem* [Ce-S]. *A compactum* X *is an* FANR *if and only if* X *is calm and movable.*

Let 2^Y be the set of all nonempty compact subsets of a metric space Y. The metric of continuity d_c is defined in [Bk1] as follows: $d_c(A,B) = \varepsilon$ if ε is the infimum of those nonnegative t such that there are t-pushes $f: A \to B$ and $g: B \to A$.

If $A,B \in 2^Y$ and there are continuous surjections $f: A \to B$ and $g: B \to A$, then the metric of continuous surjection $d_{\bar{c}}$ is defined (see [Ce2]) by: $d_{\bar{c}}(A,B) = \varepsilon$ if ε is the infimum of those nonnegative t such that there are surjective t-pushes $f: A \to B$ and $g: B \to A$.

The Hilbert cube is denoted by Q.

3. Shape Properties of Certain Maps

Let X and Y be compacta in AR-spaces M and N, respectively. Let us recall the definitions of *quasi-domination* and *quasi-equivalence* [Bk2]: X quasi-dominates Y ($X \geq_q Y$) if for every neighborhood U of Y in N there exists a neighborhood V of Y in N, $V \subset U$, and fundamental sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$, $\underline{g} = \{g_k, Y, X\}_{N,M}$ such that for almost all k, $f_k g_k | V \simeq i_{V,U}$ in U, where $i_{V,U}$ is the inclusion of V into U.

If for every neighborhood (W,U) of (X,Y) in (M,N) there exist neighborhoods (W_1, V) of (X,Y) in (M,N), $W_1 \subset W$, $V \subset U$, and fundamental sequences $\underline{f}$, $\underline{g}$ as above with $f_k g_k | V \simeq i_{V,U}$ in U and $g_k f_k | W_1 \simeq i_{W_1,W}$ in W for almost all k, then X and Y are quasi-equivalent ($X \simeq_q Y$). These notions are in general weaker than shape domination and shape equivalence, respectively, but they coincide when Y is calm (when X and Y are calm, respectively) [Bx2].

(3.1) *Theorem. Let* X *and* Y *be compacta and let* $f: X \to Y$ *be* ARI. *Then* $X \geq_q Y$. *If* f *is* AI, *then* $X \simeq_q Y$.

Proof. There is no loss of generality in assuming M = N = Q [Bk2]. Let U be a neighborhood of Y in Q. There is a compact ANR neighborhood V′ of Y in Q, $V' \subset U$, and an $\varepsilon > 0$ such that ε-close maps into V′ are homotopic. Since f is ARI, there exist $g_\varepsilon: Y \to X$ such that fg_ε is an ε-push. Let $\underline{f}$, $\underline{g}$ be fundamental sequences generated by f and g_ε, respectively. There is a neighborhood V of Y in Q, $V \subset V'$, such that $f_k g_k | V$ is an ε-push and $f_k g_k(V) \subset V'$ for almost all k. By choice of ε, $f_k g_k | V \simeq i_{V,U}$ in V′, hence in U, for almost all k. Thus $X \geq_q Y$.

If f is AI, then for a neighborhood (W,U) of (X,Y) in (Q,Q) there exist compact ANR neighborhoods W', V' of X and Y in Q, respectively, $W' \subset W$, $V' \subset U$, and an $\varepsilon > 0$ such that ε-close maps into either of W' or V' are homotopic. Since f is AI, there exist $g_\varepsilon: Y \to X$ and $f_\varepsilon: X \to Y$ such that fg_ε is $(\varepsilon/2)$-close to 1_Y, $g_\varepsilon f_\varepsilon$ is δ-close to 1_X, where $0 < \delta < \varepsilon$ and $d(x_1,x_2) < \delta$ implies $d(f(x_1),f(x_2)) < \varepsilon/2$. Therefore, for $x \in X$,

$$d(f(x),f_\varepsilon(x)) \leq d(f(x),fg_\varepsilon f_\varepsilon(x)) + d(fg_\varepsilon f_\varepsilon(x), f_\varepsilon(x)) < \text{(by choice of } \delta\text{) } \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus f and f_ε are ε-close.

If $\underline{f}$, $\underline{g}$, and $\underline{F}$ are fundamental sequences generated by f, g_ε, and f_ε, respectively, it follows from our choice of ε that there is a neighborhood(W_1,V) of (X,Y) in (Q,Q), $W_1 \subset W'$, $V \subset V'$, such that $F_k g_k|W_1 \simeq f_k g_k|W_1 \simeq i_{W_1,W}$ in W', hence in W, and (by our choice of δ) such that $g_k F_k|V \simeq i_{V,U}$ in V', hence in U.

It follows that $X \simeq_q Y$.

(3.2) *Corollary. Let* X *and* Y *be compacta,* f: X → Y *an* ARI *map. Then*

a) *if* Y *is calm,* Sh X $\geq$ Sh Y.

b) *if* f *is* AI *and* X *is calm,* Sh X $\leq$ Sh Y.

c) *if* f *is* AI *and* X *and* Y *are both calm,* Sh X = Sh Y.

Proof. This follows immediately from (3.1) and [Bx2, (3.3)].

The following is suggested by [F-R, 3.4] and gives a partial answer to [F-R, Question 5]:

(3.3) *Corollary. Let* $X \in \text{SAANR}_C$ *and let* $r: X \to Y$ *be a refinable map.*

a) *If* $X \in \text{SAANR}_N$, $\text{Sh } Y \geq \text{Sh } X$.

b) *If* $Y \in \text{SAANR}_N$, $\text{Sh } X \geq \text{Sh } Y$.

c) *If* $X, Y \in \text{SAANR}_N$, $\text{Sh } X = \text{Sh } Y$.

Proof. We have $X \in \text{SAANR}_N$ if and only if $X \in \text{SAANR}_C$ and $X \in \text{FANR}$ [Ce2], $X \in \text{AANR}_C$ implies X is movable [Bg, Theorem 6], and (2.1) imply: $X \in \text{SAANR}_N$ if and only if $X \in \text{SAANR}_C$ and X is calm.

It follows from [Ce3, opening remarks in §5] that f is AI. The assertions follow from (3.2).

We remark that it follows that Y in (3.3) is an SAANR_C, by [P2, Theorem 2].

For the collection of ARI maps that are strongly approximately right invertible (SARI) and for the collection of AI maps that are strongly approximately invertible (SAI) (see [Ce3] for definitions) we have:

(3.4) *Corollary. Let* $f: X \to Y$ *be a map between compacta.*

a) *If* $X \in \text{AANR}_N$ *and* f *is* SARI, $\text{Sh } X \geq \text{Sh } Y$.

b) *If one of* X *or* Y *is an* AANR_N *and* f *is* SAI, $\text{Sh } X = \text{Sh } Y$.

Proof. We use the fact that $X \in \text{AANR}_N$ implies $X \in \text{FANR}$ [Gm], and therefore by (2.1) X is calm.

a) We have $Y \in \text{AANR}_N$ [Ce3, (5.2a)], hence Y is calm. The assertion follows from (3.2a).

b) We have both $X, Y \in \text{AANR}_N$ [Ce3, (5.2b)]. The assertion follows from (3.2c).

4. Some Function Spaces and Hyperspaces

In this section we assume X is a compactum, Y a metric space, and Y^X is the space of maps from X to Y with the compact-open (=sup-metric) topology. Suppose $\{f_i\}_{i=0}^{\infty} \subset Y^X$ with $A_i = f_i(X)$ for all i. What does $\lim_{i\to\infty} f_i = f_0$ imply about $\{A_i\}_{i=0}^{\infty}$ with respect to hyperspaces? It is clear that $\lim_{i\to\infty} A_i = A_0$ in the topology of the Hausdorff metric. Borsuk's example [B1] of arcs converging to S^1 in the Hausdorff metric but not d_c may be used to construct a convergent sequence $f_i \to f_0$ in $(S^1)^I$ such that $A_0 \neq \lim_{i\to\infty} A_i$ in d_c. The approach of [Bx1], that by restricting X or by considering appropriate subspaces of Y^X we may obtain interesting results, is used here.

Let us define R(X,Y) and ARI(X,Y) to be the subspaces of Y^X consisting of those maps f such that $f: X \to f(X)$ is refinable or ARI, respectively. Let d^s be the sup-metric for Y^X.

(4.1) *Theorem. Let* X, Y, $\{f_i\}_{i=0}^{\infty}$, $\{A_i\}_{i=0}^{\infty}$ *be as above. Suppose* $X \in SAANR_C$ *and* $f_i \in R(X,Y)$ *for* $i > 0$. *If* $f_0 \in R(X,Y)$ *then* $\lim_{i\to\infty} d_{\overline{C}}(A_i, A_0) = 0$.

Proof. Suppose $f_0 \in R(X,Y)$. Let $\varepsilon > 0$. Since $X \in SAANR_C$, there are [P2, Theorem 1] continuous surjections $g_i: A_i \to X$ such that $f_i g_i: A_i \to A_i$ is an $(\varepsilon/2)$-push for all i.

Fix i such that $d^s(f_i, f_0) < \varepsilon/2$.

Consider the continuous surjections $F_i = f_0 g_i: A_i \to A_0$, $G_i = f_i g_0: A_0 \to A_i$. We have $d^s(F_i, 1_{A_i}) \leq d^s(f_0 g_i, f_i g_i) + d^s(f_i g_i, 1_{A_i}) < \varepsilon/2 + \varepsilon/2 = \varepsilon$, and $d^s(G_i, 1_{A_0}) \leq$

$d^s(f_i g_o, f_o g_o) + d^s(f_o g_o, 1_{A_o}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$

It follows that $\lim_{i\to\infty} d_{\bar{c}}(A_i, A_o) = 0$.

(4.2) *Theorem. Suppose* $\lim_{i\to\infty} f_i = f_o$ *in* Y^X *with* $f_i \in ARI(X,Y)$ *for* $i > 0$. *Then* $f_o \in ARI(X,Y)$ *if and only if* $\lim_{i\to\infty} d_c(A_i, A_o) = 0$.

Proof. Suppose $f_o \in ARI(X,Y)$. Let $\varepsilon > 0$. Fix i such that $d^s(f_i, f_o) < \varepsilon/2$. There exist maps $h_i\colon A_i \to X$, $h_o\colon A_o \to X$ such that $f_i h_i$ and $f_o h_o$ are $(\varepsilon/2)$-pushes. Consider $F = f_o h_i\colon A_i \to A_o$, $G = f_i h_o\colon A_o \to A_i$.

We have

$$d^s(F, 1_{A_i}) \le d^s(f_o h_i, f_i h_i) + d^s(f_i h_i, 1_{A_i}) < \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{and}$$

$$d^s(G, 1_{A_o}) \le d^s(f_i h_o, f_o h_o) + d^s(f_o h_o, 1_{A_o}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It follows that $\lim_{i\to\infty} d_c(A_i, A_o) = 0$.

Conversely, if $\lim_{i\to\infty} d_c(A_i, A_o) = 0$, fix $\varepsilon > 0$ and i such that $d_c(A_i, A_o) < \varepsilon/3$ and $d^s(f_i, f_o) < \varepsilon/3$. There exist maps $g_i\colon A_o \to A_i$, $h_i\colon A_i \to X$ such that g_i and $f_i h_i$ are $(\varepsilon/3)$-pushes. Consider $h = h_i g_i\colon A_o \to X$. We have

$$d^s(f_o h, 1_{A_o}) \le d^s(f_o h_i g_i, f_i h_i g_i) + d^s(f_i h_i g_i, g_i) + d^s(g_i, 1_{A_o}) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

It follows that $f_o \in ARI(X,Y)$.

References

[Bg] S. A. Bogatyi, *Approximate and fundamental retracts*, Mat. Sbornik 93 (135) (1974), 91-103.

[Bk1] K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fund. Math. 41 (1954), 168-202.

[Bk2] ________, *Some quantitative properties of shapes*, Fund. Math. 93 (1976), 197-212.

[Bx1] L. Boxer, *Retraction spaces and the homotopy metric*, Topology and Its Appl. 11 (1980), 17-29.

[Bx2] ________, *Remarks on quasi-domination* (preprint).

[Ce1] Z. Čerin, *Homotopy properties of locally compact spaces at infinity--calmness and smoothness*, Pacific J. of Math. 79 (1978), 69-91.

[Ce2] ________, *Surjective approximate absolute (neighborhood) retracts*, Top. Proc. 6 (1981), 5-24.

[Ce3] ________, $\mathcal{C}_p$-e-*movable and* $\mathcal{C}$-e-*calm compacta and their images*, Compositio Math. (to appear).

[Ce-S] ________ and A. P. Šostak, *Some remarks on Borsuk's fundamental metric*, in Topology, v. 1 (Á. Császár, ed.), 233-252, North-Holland, New York, 1980.

[Cl] M. Clapp, *On a generalization of absolute neighborhood retracts*, Fund. Math. 70 (1971), 117-130.

[F-R] J. Ford and J. W. Rogers, Jr., *Refinable maps*, Colloquium Math. 39 (1978), 263-269.

[G] R. Geoghegan, ed., *Open problems in infinite-dimensional topology*, Top. Proc. 4 (1979), 287-338.

[Gm] A. Gmurczyk, *Approximative retracts and fundamental retracts*, Colloquium Math. 23 (1971), 61-63.

[N] H. Noguchi, *A generalization of absolute neighborhood retracts*, Kodai Math. Sem. Reports 1 (1953), 20-22.

[P1] P. Patten, *Locally connected generalized absolute neighborhood retracts*, Top. Proc. 3 (1978), 159-168.

[P2] ________, *Refinable maps and generalized absolute neighborhood retracts* (preprint).

Niagara University

Niagara University, New York 14109