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by

J. Dydak and J. Segal

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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In recent years a number of generalizations of the notion of absolute neighborhood retract (ANR) have appeared. Among these are approximate absolute neighborhood retracts (AANR) due to M. H. Clapp [4] and the equivalent notion of NE-sets due to K. Borsuk [3]. S. Mardešić [11] has recently given a generalization called approximate polyhedra (AP) which applies to topological spaces. Mardešić's AP's agree with AANR's in the compact metric case so for brevity we use his notation throughout this paper. Although the AP's form a much larger class than ANR's, Clapp showed that they possess many of the fixed point properties of ANR's. Moreover, he showed that the AP's form a quite natural class in the sense that they are precisely the limits of polyhedra in the metric of continuity.

In shape theory Borsuk [2] generalized the ANR's with the notion of fundamental absolute neighborhood retracts (FANR's). While this class possesses many of the desirable shape analogues of ANR's, it also has members with considerable local pathology. Not surprisingly, in shape theory base points cause considerable difficulty (see R. Geoghegan [9]). However, recently H. M. Hastings and A. Heller [10] have shown that every FANR is a pointed FANR. Whether this also holds for Borsuk's even broader generalization of ANR's, namely movable continua, is still not known. In other words, is every movable continuum also pointed movable? In this paper we show that AP's with the fixed point property (a subclass of movable continua) are pointed movable. We also show that every regularly movable continuum has the shape of an AP and that a compactum is movable iff it is shape dominated by an AP. In this paper compactum means compact metric and continuum means connected compactum.

Definition 1. (Clapp) A compactum X is an *approximate absolute neighborhood retract* (AANR) provided when X is embedded in a metric space M, then for every $\varepsilon > 0$ there exist a neighborhood U of X in M and a map $r: U \to X$ such that the distance $d(r(x),x) < \varepsilon$ for all x in X.

Definition 2. (Mardešić) A compactum X is an *approximate polyhedron* (AP) if for each $\varepsilon > 0$ there is a polyhedron P and maps $f: X \to P$, $g: P \to X$ such that the distance $d(gf(x),x) < \varepsilon$ for all x in X.

Remark 1. Mardešić [11] actually defines AP's for the class of topological spaces by using normal coverings instead of the metric notion of "$\varepsilon > 0$." However, he shows that compact metric AP's agree with AANR's. Moreover, he proves that if a paracompactum X is LC^{n-1} and of covering dimension $\dim X \leq n < \infty$, then X is an AP.

Theorem 1. *A compactum* X *is movable iff* sh X $\leq$ sh Y *where* Y *is an* AP.

Proof. Assume that X is movable. Then C. Cox [5] and S. Spież [13] have shown (see also S. A. Bogatyĭ [1]) that if X is a subcompactum of a compactum M and X is the intersection of open-closed subsets of M, then sh X $\leq$ sh M. Let $X = \lim\{X_n, p_n^{n+1}\}$ be the inverse limit of the ANR-sequence $\{X_n, p_n^{n+1}\}$. We take M to be a space X* defined as the disjoint union of X and all the X_n. A basis for the topology of X* consists of all the open sets U_n from X_n and of the sets

$$U_n^* = \bigcup_{n \leq n'} p_{nn'}^{-1}(U_n) \cup p_n^{-1}(U_n).$$

For every positive integer n we also define a map $p_n^*: X_n^* \to X_n$, where

$$X_n^* = \bigcup_{n \leq n'} X_{n'} \cup X \subset X^*,$$

by putting $p_n^*|X = p_n, p_n^*|X_{n'} = p_{nn'}, n \leq n'$. Then $X = \cap X_n^*$ where the X_n^* are open-closed subsets of X*, so that sh X $\leq$ sh X* (see Mardešić and Segal [12, p. 48]). Moreover, X* is an AP since for every $\varepsilon > 0$ there is a polyhedron, namely $X_n^+ = X_1 + X_2 + \cdots + X_n$ (disjoint union) for n sufficiently large, and maps $f: X^* \to X_n^+$ (defined by $f(x) = p_n^*(x)$ for x in X_n^* and $f(x) = x$ for x in X_m, $m < n$) and $g: X_n^+ \to X^*$ (defined as the inclusion) such that $d(gf(x), x) < \varepsilon$, for all x in X*. Thus we have X* is an AP which shape dominates X.

The converse is proved by Bogatyĭ [1, Theorem 6]. Actually he proves the stronger statement that an AP is internally movable.

Definition 3. A compactum X is said to be *regularly movable* provided there exists an ANR-sequence $\underline{X} = \{X_n, p_n^{n+1}\}$ such that $X = \varprojlim \underline{X}$ and each bonding map p_n^{n+1} is a homotopy domination. So $p_{n\#}^{n+1}: \pi_1(X_{n+1}) \to \pi_1(X_n)$ is epic and therefore pro-$\pi_1(X)$ is Mittag-Leffler. This implies that X is pointed 1-movable which implies it is pointed movable.

Theorem 2. *If* X *is regularly movable, then there exists an* AP Y *with* sh X = sh Y.

Proof. Let $X = \underleftarrow{\lim}(X_n, p_n^{n+1})$, where the p_n^{n+1} are homotopy dominations and the X_n are ANR's. Define inductively $Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_n \in$ ANR and retractions $r_n: Y_{n+1} \to Y_n$ such that $Y_n \supset X_n$ and the following diagram

$$\begin{array}{ccc} X_{n+1} & \hookrightarrow & Y_{n+1} \\ {\scriptstyle p_n^{n+1}} \downarrow & & \downarrow {\scriptstyle r_n} \\ X_n & \hookrightarrow & Y_n \end{array}$$

is homotopy commutative and X_n is a deformation retract of Y_n. Let $Y_1 = X_1$. Suppose $Y_1, Y_2, \cdots, Y_n$ are defined. Let $g_n: X_n \to X_{n+1}$ satisfy $p_n^{n+1} g_n \simeq id_{X_n}$. Then $Y_{n+1} = M(\tilde{g}_n)$, the mapping cylinder of an extension $\tilde{g}_n: Y_n \to X_{n+1}$ of g_n. Since $\tilde{g}_n$ has a left homotopy inverse, there exists a retraction $r_n: Y_{n+1} \to Y_n$ possessing the desired properties.

It is easy to see that any $Y = \underleftarrow{\lim}(Y_n, q_n^{n+1})$ such that the q_n^{n+1} are retractions, is an AP.

Remark 2. D. A. Edwards and R. Geoghegan [8] showed that there are compacta shape dominated by ANR's (i.e. FANR's) which fail to have the shape of a compact ANR. However, J. Dydak and A. Trybulec [7] showed if X is regularly movable and shape dominated by an ANR, then X has the shape of a compact ANR.

Lemma 1. *Suppose* X *is a subcontinuum of the Hilbert cube. If for each* $\varepsilon > 0$ *there exists a neighborhood* U_ε *of* X *in* Q *and a map* $r_\varepsilon: U_\varepsilon \to X$ *such that* $\rho(r_\varepsilon(x), x) < \varepsilon$ *for* $x \in X$ *and* $r_\varepsilon(x_\varepsilon) = x_\varepsilon$ *for some* $x_\varepsilon \in X$, *then* X *is pointed movable.*

Proof. The assumptions on X imply that X is movable. Since a movable and pointed 1-movable continuum is pointed movable we need only show that X is pointed 1-movable. So take $x_0 \in X$ and let U be a neighborhood of X in Q. Then, for sufficiently small ε, the map r_ε is homotopic rel. x_ε in U to the inclusion map $U_\varepsilon \hookrightarrow U$. Suppose W is a neighborhood of X in U_ε and α is a loop in U_ε at x_0. Take a path β joining x_0 and x_ε in W. Since $(U_\varepsilon, x_\varepsilon) \hookrightarrow (U, x_\varepsilon)$ is homotopic to $r_\varepsilon: (U_\varepsilon, x_\varepsilon) \to (U, x_\varepsilon)$, the loop $\gamma = r_\varepsilon \cdot (\beta^{-1}\alpha\beta)$ is in W and is homotopic rel. x_ε to $\beta^{-1}\alpha\beta$ in U. Hence $\beta\gamma\beta^{-1}$ is a loop at x_0 in W homotopic rel. x_0 to α in U. Thus X is pointed 1-movable and consequently X is pointed movable.

Corollary 1. *If* X *is an* AP *with the fixed point property, then* X *is pointed movable.*

Definition 4. By *generalized Euler characteristic* we mean $\check{\chi}(x) = \sum_{i=0}^{\infty} (-1)^i \cdot \operatorname{rank} \check{H}_i(X)$ which is defined for a compactum X such that $\check{H}_i(X)$ is finitely generated for all i and is trivial for almost all i.

Theorem 3. *If* X ∈ AP *and* $\check{\chi}(X) \neq 0$, *then* X *is pointed movable.*

Proof. Embed X in the Hilbert cube Q and take a decreasing sequence $(U_n)_{n=1}^{\infty}$ of compact ANR's with $X = \bigcap_{n=1}^{\infty} U_n$. Then $\check{H}_i(X) = \varprojlim H_i(U_n)$. Fix i > 0. We are going to prove that the natural homomorphism $\alpha_n: \check{H}_i(X) \to H_i(U_n)$ is a monomorphism for n sufficiently large. Let G_n = image of α_n. Then $\check{H}_i(X) = \varprojlim G_n$, each G_n is finitely generated and (G_n) satisfies the Mittag-Leffler condition (see Dydak and Segal [6, Lemma 6.1.5 and Theorem 6.1.7 on p. 78]). Hence (G_n) is stable (see Theorem 6.1.8 on p. 80 in Dydak and Segal [6]) and α_n must be a monomorphism for n sufficiently large. Since almost all groups $\check{H}_i(X)$ are trivial, there is a neighborhood U of X such that the natural homomorphism $\beta_i: \check{H}_i(X) \to H_i(U)$ is a monomorphism for all i. Let $\varepsilon > 0$ and take $r_\varepsilon: U_\varepsilon \to X$ such that $\rho(r_\varepsilon(x),x) < \varepsilon$ for $x \in X$ and $r_\varepsilon|X: X \to X$ is homotopic in U to the inclusion $X \hookrightarrow U$. Then $\beta_i \cdot \check{H}_i(r_\varepsilon|X) = \beta_i = \beta_i \cdot \check{H}_i(id_X)$ and therefore $\check{H}_i(r_\varepsilon|X) = \check{H}_i(id_X)$, since β_i is a monomorphism. Consequently, the generalized Lefschetz number $\lambda(r_\varepsilon|X)$ of $r_\varepsilon|X$ is equal to $\check{\chi}(X) = \lambda(id_X)$ and, by Clapp's generalization of the Lefschetz fixed point theorem, there exists $x_\varepsilon \in X$ with $r_\varepsilon(x_\varepsilon) = x_\varepsilon$. By the Lemma, X is pointed movable.

Problem 1. Is every continuum X ∈ AP pointed movable?

Problem 2. Does every X ∈ AP have the shape of a regularly movable continuum?

Problem 3. Suppose X ∈ FANR ∩ AP. Is there a finite CW complex of the same shape as X?

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University of Warsaw
Warsaw, Poland
and
University of Washington
Seattle, Washington 98195