TOPOLOGY PROCEEDINGS

Volume 6, 1981

Pages 287–298

http://topology.auburn.edu/tp/

SEQUENTIAL ORDER OF HOMOGENEOUS AND PRODUCT SPACES

by

L. Foged

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

SEQUENTIAL ORDER OF HOMOGENEOUS AND PRODUCT SPACES

L. Foged

In this paper we will present some examples of sequential spaces with properties related to their sequential order [1]. In section 1 we will show that homogeneous spaces may have any prescribed sequential order. In section 2 we will show that the sequential order of the sequential coreflection of a product of spaces of "small" sequential order may have "large" sequential order and will answer a question posed by Michael [4] when we give an example of a sequential $\aleph_0$-space Z so that the sequential coreflection of Z^2 is not regular. Our constructions use sets of sequences as underlying sets and will be facilitated by the following notation. If X is a set, we denote finite sequences in X by $\langle x_0, x_1, \cdots, x_k \rangle$, where $k \in \omega$ and every $x_j \in X$. If $\{x_0, x_1, \cdots, x_k\} \subset X$ and $S \subset X$, we let $\langle x_0, x_1, \cdots, x_k, S \rangle = \{\langle x_0, x_1, \cdots, x_k, x_{k+1} \rangle : x_{k+1} \in S\}$ and let $\langle x_0, x_1, \cdots, x_k, S, \cdots \rangle$ be the set of all finite sequences in X which extend a member of $\langle x_0, x_1, \cdots, x_k, S \rangle$.

1. Homogeneous Spaces

The authors of [1] asked whether their space S_ω is the only countable, Hausdorff, homogeneous, sequential space which is not first countable. A technique which produces many spaces with these properties was given in [3]; this technique produces spaces which, like S_ω, have sequential order ω_1. While [5] gives a non-regular example with the listed properties and has sequential order 2, the situation among regular spaces is clarified by the following.

A. *Examples*. For every $\alpha \leq \omega_1$ there is a countable, regular, homogeneous, weakly first countable space X_α with sequential order α.

Following [1], if A is a subset of a topological space X then $A^0 = A$; if $\alpha = \beta+1$, then A^α = the set of limits of sequences in A^β; if α is a limit ordinal, then $A^\alpha = \cup_{\beta<\alpha} A^\beta$.

We will construct our spaces by induction on α, letting X_0 be a countable discrete space. Suppose we have constructed X_β for all $\beta < \alpha$.

1. α *is not a limit ordinal.*

Write $\alpha = \beta+1$; let B be a wfc system [2] for X_β; pick a distinguished point x^* in X_β. Let X_α be the set of finite sequences $\langle x_0, x_1, \cdots, x_{2k} \rangle$ in $X_\beta \cup \omega$ so that

$$x_j \in X_\beta \text{ if } j \text{ is even,}$$
$$x_j \in \omega \text{ if } j \text{ is odd.}$$

We define a wfc system B' for X_α as follows. If $n < \omega$ and $\sigma = \langle x_0, x_1, \cdots, x_k \rangle \in X_\alpha$, then

$$B'(n,\sigma) = \{\sigma\} \cup \langle x_0, \cdots, x_{k-1}, B(n,x_k)\backslash\{x_k\}, \cdots \rangle$$
$$\cup \langle x_0, \cdots, x_k, \omega\backslash n, x^*, \cdots \rangle,$$

where $\omega\backslash n = \{n, n+1, n+2, \cdots\}$.

We may then show that the sets

$$\{\sigma\} \cup \langle x_0, x_1, \cdots, x_{k-1}, U\backslash\{x_k\}, \cdots \rangle \cup$$

$$\bigcup_{n \geq n_0} \langle x_0, x_1, \cdots, x_k, n, V_n, \cdots \rangle$$

where $\sigma = \langle x_o, x_1, \cdots, x_k \rangle \in X_\alpha$, U is a neighborhood of x_k in X_β, $n_o < \omega$, and every V_n is a neighborhood of x^* in X_β, make up a local neighborhood base at σ.

Inductively, X_α may be shown to have a base of clopen sets.

For each $\sigma = \langle x_o, x_1, \cdots, x_k \rangle \in X_\alpha$ the clopen neighborhood $\langle x_o, x_1, \cdots, x_{k-1}, X_\beta, \cdots \rangle$ of σ is canonically homeomorphic to $X_\alpha = \langle X_\beta, \cdots \rangle$ under a homeomorphism carrying σ to $\langle x^* \rangle$. It follows that distinct points σ and τ lie in disjoint clopen sets which admit a homeomorphism carrying σ to τ; hence X_α is homogeneous.

We wish to show that X_α has sequential order α. Suppose $A \subset X_\alpha$ and $\langle x^* \rangle \in \text{cl}A \setminus A$. Then as $\{\langle x^* \rangle\} \cup \langle X_\beta \setminus \{x^*\}, \cdots \rangle \cup \langle x^*, \omega, X_\beta, \cdots \rangle$ is a neighborhood of $\langle x^* \rangle$, we may assume that either (1) $A \subset \langle X_\beta \setminus \{x^*\}, \cdots \rangle$, or (2) $A \subset \langle x^*, \omega, X_\beta, \cdots \rangle$.

(case 1) $A \subset \langle X_\beta \setminus \{x^*\}, \cdots \rangle$. Let $\pi: \langle X_\beta \setminus \{x^*\}, \cdots \rangle \to \langle X_\beta \setminus \{x^*\} \rangle$ be the natural "trimming" function. Since $\langle x^* \rangle \in \text{cl } A$, we may use the given neighborhood base for $\langle x^* \rangle$ to show that $\langle x^* \rangle \in \text{cl } \pi(A)$. As $\pi(A) \subset \langle X_\beta \rangle$ and $\langle X_\beta \rangle$ is closed and homeomorphic to X_β, we deduce that $\langle x^* \rangle \in [\pi(A)]^\beta$. That $\langle x^* \rangle \in A^\beta$, follows from (*).

(*) For every γ $[\pi(A)]^\gamma \setminus \pi(A) \subset A^\gamma$.

If $\gamma = 1$ and $\langle x_o \rangle \in \text{seq cl } \pi(A) \setminus \pi(A)$, then there is a sequence $\langle \sigma_n \rangle_{n<\omega}$ in A so that $\langle \pi(\sigma_n) \rangle_{n<\omega}$ converges to $\langle x_o \rangle$. Recalling the definition of B', we see that in fact $\langle \sigma_n \rangle_{n<\omega}$ converges to $\langle x_o \rangle$. Assume now that (*) holds for all $\delta < \gamma$. If $\gamma = \delta+1$, let $\langle x_o \rangle \in [\pi(A)]^\gamma \setminus \pi(A)$; we may assume $\langle x_o \rangle \notin [\pi(A)]^\delta$. Then there is a sequence $\langle \sigma_n \rangle_{n<\omega}$ in $[\pi(A)]^\delta \setminus \pi(A)$ converging to $\langle x_o \rangle$, the induction hypothesis yielding that $\langle x_o \rangle \in A^\gamma$. On the other hand, if γ is a limit ordinal, then $[\pi(A)]^\gamma \setminus \pi(A) = \cup_{\delta<\gamma}[\pi(A)]^\delta \setminus \pi(A) \subset \cup_{\delta<\gamma} A^\delta = A^\gamma$. This establishes (*).

(case 2) $A \subset \langle x^*, \omega, X_\beta, \cdots \rangle$. Let $\pi\colon \langle x^*, \omega, X_\beta, \cdots \rangle \to \langle x^*, \omega, X_\beta \rangle$ be the trimming function. Again, $\langle x^* \rangle \in \mathrm{cl}\ A$ implies that $\langle x^* \rangle \in \mathrm{cl}\ \pi(A)$. Because $\pi(A) \subset \langle x^*, \omega, X_\beta \rangle \cup \{\langle x^* \rangle\}$ and because the latter set is closed and homeomorphic to the sequential sum [1] of $\aleph_o$ copies of X_β (thus has sequential order $\beta + 1$), we get $\langle x^* \rangle \in [\pi(A)]^{\beta+1}$. We may verify that (*) holds for π, and thus $\langle x^* \rangle \in A^{\beta+1}$.

Thus in either case $\langle x^* \rangle \in A^{\beta+1} = A^\alpha$. Hence the sequential order of X_α is no greater than α. As X_α contains a closed copy of the sequential sum of $\aleph_o$ copies of X_β, the sequential order is precisely α.

2. *α is a limit ordinal.*

Write $\alpha = \sup\{\beta_i\colon i < \omega\}$ so that the β_i's are not limit ordinals. For every $i < \omega$, let $X_i = X_{\beta_i}$; distinguish a point x^i in X_i, and let $X_i^* = X_i \setminus \{x^i\}$. X_α is the set of all finite sequences $\langle x_o, x_1, \cdots, x_k \rangle$ in $\cup_{i<\omega} X_i^*$ so that for all $j < k$

$$\text{if } x_j \in X_i^*, \text{ then } x_{j+1} \notin X_i^*.$$

Let B_i be a wfc system for X_i so that for every $x \in X_i$ and every $n \leq i$, $B_i(n,x) = X_i$. Now define a wfc system B for X_α as follows: For $n < \omega$ and $\sigma = \langle x_o, x_1, \cdots, x_k \rangle \in X_\alpha$ with $x_k \in X_{i_o}^*$ let

$$B(n,\sigma) = \{\sigma\} \cup \langle x_o,x_1,\cdots,x_{k-1},B_{i_o}(n,x_k)\backslash\{x_k\},\cdots\rangle$$
$$\cup \cup_{i\neq i_o}\langle x_o,x_1,\cdots,x_k,B_i(n,x^i)\backslash\{x^i\},\cdots\rangle.$$

A neighborhood base at σ is formed by the sets of the form

$$\{\sigma\} \cup \langle x_o,x_1,\cdots,x_{k-1},U\backslash\{x_k\},\cdots\rangle \cup$$
$$\cup_{i\neq i_o}\langle x_o,x_1,\cdots,x_k,V^i\backslash\{x^i\},\cdots\rangle,$$

where U is a neighborhood of x_k in X_{i_o}, for all $i \neq i_o$ V^i is a neighborhood in X_i of x^i, and for all but finitely many i $V^i = X^i$. X_α may be shown to have a base of clopen sets.

Fix $x_o' \in X_o^*$. Let $\sigma = \langle x_o,x_1,\cdots,x_k\rangle \in X_\alpha$ with $x_k \in X_i^*$. We will find clopen neighborhoods of $\langle x_o'\rangle$ and σ that are homeomorphic under a mapping carrying σ to $\langle x_o'\rangle$, giving homogeneity as before. If $i = 0$, then we can find a homeomorphism f on X_o so that $f(x_k) = x_o'$ and a clopen neighborhood V of x_k in X_o so that $x^o \not\in V \cup f(V)$; thus $\langle f(V),\cdots\rangle$ and $\langle x_o,x_1,\cdots,x_{k-1},V,\cdots\rangle$ are the desired clopen neighborhoods. If on the other hand $i \neq 0$, find homeomorphisms f_o on X_o and f_i on X_i so that $f_o(x^o) = x_o'$ and $f_i(x_k) = x^i$; also find clopen neighborhoods V_o of x^o and V_i of x_k so that $x^o \not\in f_o(V_o)$ and $x^i \not\in V_i$. The natural map defined piecewise from the clopen neighborhood $\{\sigma\} \cup \langle x_o,x_1,\cdots,x_{k-1}, V_i\backslash\{x_k\},\cdots\rangle \cup \langle x_o,x_1,\cdots,x_k,V_o\backslash\{x^o\},\cdots\rangle \cup \cup_{j\neq i,0} \langle x_o,x_1,\cdots,x_k,X_j^*,\cdots\rangle$ of σ onto the clopen neighborhood $\{\langle x_o'\rangle\} \cup \langle x_o',f_i(V_i)\backslash\{x^i\},\cdots\rangle \cup \langle f_o(V_o)\backslash\{x_o'\},\cdots\rangle \cup \cup_{j\neq i,0} \langle x_o',X_j^*,\cdots\rangle$ of $\langle x_o'\rangle$ is a homeomorphism.

Suppose $A \subset X_\alpha$ and $\langle x_o'\rangle \in \text{cl } A\backslash\text{seq cl } A$. Since

$\langle x_o' \rangle \notin$ seq cl A, there is an $i_o < \omega$ so that A misses $\cup_{i>i_o} \langle x_o', X_i^*, \cdots \rangle$. We may assume that either $A \subset \langle X_o^* \setminus \{x_o'\}, \cdots \rangle$ or that $A \subset \langle x_o', X_i^*, \cdots \rangle$ for some $i \leq i_o$. If $A \subset \langle x_o', X_i^*, \cdots \rangle$, let $\pi: \langle x_o', X_i^*, \cdots \rangle \to \langle x_o', X_i^* \rangle$, be the trimming function. Then $\langle x_o' \rangle \in \mathrm{cl}\pi(A)$, and since $\pi(A) \subset \langle x_o', X_i^* \rangle$ and $\langle x_o', X_i^* \rangle \cup \{\langle x_o' \rangle\}$ is closed and homeomorphic to X_i, we have that $\langle x_o' \rangle \in [\pi(A)]^{\beta_i} \subset [\pi(A)]^{\alpha}$. One may show that π satisfies (*); thus $\langle x_o' \rangle \in A^{\alpha}$. The proof in the case $A \subset \langle X_o \setminus \{x_o'\}, \cdots \rangle$ is similar. Hence X_α has sequential order no larger than α. The subsets $\{\langle x_o \rangle\} \cup \langle x_o, X_i^* \rangle$ of X_α are closed and homeomorphic to X_i when $x_o \notin X_i^*$, so the sequential order of X_α is at least $\sup_{i<\omega} \beta_i = \alpha$. This completes the proof.

2. Product Spaces

While the product of two sequential spaces need not be sequential, if X and Y are sequential there is a natural sequential space topology for the set $X \times Y$: decree that a sequence $\langle (x_n, y_n) \rangle_{n\in\omega}$ "converges" to (x,y) if and only if $\langle x_n \rangle_{n\in\omega}$ converges to x in X and $\langle y_n \rangle_{n\in\omega}$ converges to y in Y; define U to be open in $X \times Y$ if it is sequentially open with respect to these "convergent" sequences. This is the sequential coreflection of the usual product topology, denoted henceforth by $\sigma(X \times Y)$.

Our next examples show that there is no natural bound on the sequential order of $\sigma(X \times Y)$ based on the sequential orders of X and Y.

B. *Examples*. There are countable regular spaces X and Y so that X is Fréchet and Y is weakly first countable with sequential order 2 so that σX^2 and σY^2 have sequential order ω_1.

Let X be the set of all finite sequences in ω of even (possibly 0) length. A set U is open in X if and only if for every $\langle n_1,n_2,\cdots,n_{2k}\rangle \in U$ and every $i \in \omega$, there is a $j \in \omega$ so that $\langle n_1,n_2,\cdots,n_{2k},i,\omega\backslash j,\cdots\rangle \subset U$.

The space σX^2 contains a closed copy of S_ω. Let $s(\phi) = (\phi,\phi)$ and for $n_1 \in \omega$ let $s(n_1) = (\langle 0,n_1\rangle,\phi)$. Generally, $s(n_1,n_2,\cdots,n_{2k}) = (\langle 0,n_1,n_2,\cdots,n_{2k-1}\rangle, \langle n_1,n_2,\cdots,n_{2k}\rangle)$ and $s(n_1,n_2,\cdots,n_{2k+1}) = (\langle 0,n_1,n_2,\cdots,n_{2k+1}\rangle, \langle n_1,n_2,\cdots,n_{2k}\rangle)$. Observe that $\langle s(n_1,n_2,\cdots,n_{k+1})\rangle_{n_{k+1}\in\omega}$ converges to $s(n_1,\cdots,n_k)$. We will show that these are essentially the only sequences in $S = \{s(\sigma)\colon \sigma$ a finite sequence in $\omega\}$ converging to a point of X^2, hence S is a sequentially closed copy of S_ω in X^2

Suppose with us that there is a sequence σ in S converging to $(\langle r_1,r_2,\cdots,r_{2k}\rangle,\langle s_1,s_2,\cdots,s_{2\ell}\rangle) \in X^2$ which is not eventually constant in either factor. Then there is such a $\sigma = \langle(\langle 0,n_1^p,n_2^p,\cdots,n_{i_p}^p\rangle,\langle n_1^p,n_2^p,\cdots,n_{j_p}^p\rangle)\rangle_{p\in\omega}$ so that $|i_p - j_p| = 1$, $i_p \geq 2k + 1$, $j_p \geq 2\ell + 2$ for all $p \in \omega$; $\{n_{2k+1}^p\colon p \in \omega\}$ and $\{n_{2\ell+2}^p\colon p \in \omega\}$ are infinite; $\{n_{2k}^p\colon p \in \omega\}$ and $\{n_{2\ell+1}^p\colon p \in \omega\}$ are finite. Now if $j \leq 2\ell$, $\langle n_j^p\rangle_{p\in\omega}$ is eventually constant $(=s_j)$, so $2k + 1 > 2\ell$; also $2\ell + 1 \neq 2k + 1$, so $2k + 1 > 2\ell + 2$. We also have that if $j \leq 2k - 1$, $\langle n_j^p\rangle_{p\in\omega}$ is eventually constant $(=r_{j+1})$, so $2\ell + 2 > 2k - 1$. With $2k \neq 2\ell + 2$, we get $2\ell + 2 > 2k + 1$, a contradiction.

So every convergent sequence in S is eventually constant in one of the factors. If σ is a sequence in S which is constant in the second factor, $\sigma \backslash (\langle 0,n_1,n_2,\cdots,n_{2k-1}\rangle, \langle n_1,n_2,\cdots,n_{2k}\rangle) = \langle (\langle 0,n_1,n_2,\cdots,n_{2k},n_{2k+1}^p\rangle, \langle n_1,n_2,\cdots,n_{2k}\rangle)\rangle_{p\in\omega}$ for some $\langle n_1,n_2,\cdots,n_{2k}\rangle \in X$. Likewise if σ is constant in the first factor, $\sigma\backslash(\langle 0,n_1,n_2,\cdots,n_{2k-1}\rangle,\langle n_1,n_2,\cdots,n_{2k-2}\rangle) = \langle (\langle 0,n_1,n_2,\cdots,n_{2k-1}\rangle,\langle n_1,n_2,\cdots,n_{2k-1},n_{2k}^p\rangle)\rangle_{p\in\omega}$, showing that every sequence in S converging to a point in X^2 is eventually constant or a subsequence of one of our canonical convergent sequences, as desired.

Let Y be the set of all non-void finite sequences of positive rationals with wfc system given by $B(m,\langle q_o,q_1,\cdots,q_k\rangle) = \langle q_o,q_1,\cdots,q_{k-1},S_m(q_k)\rangle \cup \langle q_o,q_1,\cdots,q_k,S_m(0),\cdots\rangle$, where $S_m(q) = \{r \in \mathbf{Q}^+ : |r-q| < 1/m\}$.

Now let $\langle q(j)\rangle_{j<\omega}$ be a sequence in $\mathbf{Q}^+$ converging monotonically to 0 and $\langle q(j,k)\rangle_{k<\omega}$ be a sequence in $(q(j+1),q(j)) \cap \mathbf{Q}^+$ converging monotonically to $q(j)$. We will show that the set $S = \{s(\sigma): \sigma$ a finite sequence in $\omega\}$ is a closed copy of S_ω in σY^2, where $s(\phi) = (\langle 1\rangle,\langle 1\rangle)$, $s(n_1,n_2,\cdots,n_{2k-1}) = (\langle 1,q(n_1,n_2),\cdots,q(n_{2k-3},n_{2k-2}),q(n_{2k-1})\rangle, \langle 1+q(n_1),q(n_2,n_3),\cdots,q(n_{2k-2},n_{2k-1})\rangle)$, $s(n_1,n_2,\cdots,n_{2k}) = (\langle 1,q(n_1,n_2),\cdots,q(n_{2k-1},n_{2k})\rangle,\langle 1+q(n_1),q(n_2,n_3),\cdots, q(n_{2k-2},n_{2k-1}),q(n_{2k})\rangle)$.

We will show that a sequence $\langle \sigma_p\rangle_{p<\omega} = \langle s(n_1^p,n_2^p,\cdots, n_{i_p}^p)\rangle_{p<\omega}$ in S cannot converge to a point $(\langle r_1,r_2,\cdots,r_k\rangle, \langle s_1,s_2,\cdots,s_\ell\rangle)$ of Y^2 if for infinitely many $p < \omega$ both the first coordinate $(=\sigma_p^1)$ has length $> k$ and the second coordinate $(=\sigma_p^2)$ has length $> \ell$. For if such a sequence did converge we could, by finding a subsequence, assume that the σ_p^1's extend $\langle r_1,\cdots,r_k\rangle$ and converge to 0 in the $k+1$ position, while the σ_p^2's extend $\langle s_1,\cdots,s_\ell\rangle$ and converge to 0 in the $\ell+1$ position. That is, $\{n_i^p\colon p < \omega\}$ is finite for all $i < 2k - 1$ and $\{n_{2k-1}^p\colon p < \omega\}$ is infinite, while $\{n_i^p\colon p < \omega\}$ is finite for all $i < 2\ell$ and $\{n_{2\ell}^p\colon p < \omega\}$ is infinite; this contradiction establishes our claim.

If $\langle \sigma_p\rangle_{p<\omega} = \langle s(n_1^p,n_2^p,\cdots,n_i^p)\rangle_{p<\omega}$ is a sequence in S converging to $(\langle r_1,r_2,\cdots,r_k\rangle,\langle s_1,s_2,\cdots,s_\ell\rangle)$ so that the σ_p^1's have length k, then σ_p^1 is eventually constant $(=\langle r_1,\cdots,r_{k-1}\rangle)$ in the first k-1 positions, i.e. for appropriate $n_1,n_2,\cdots,n_{2k-4}$ and large p $\langle 1,q(n_1^p,n_2^p),\cdots,q(n_{2k-5}^p,n_{2k-4}^p)\rangle = \langle 1,q(n_1,n_2),\cdots,q(n_{2k-5},n_{2k-4})\rangle$. Further, $\langle \sigma_p^1\rangle_{p<\omega}$ converges to $r_k \neq 0$ in the k position, so that $\langle n_{2k-3}^p\rangle_{p<\omega}$ is eventually constant $(=n_{2k-3})$ and $\{n_{2k-2}^p\colon p < \omega\}$ is infinite. Consequently $\langle \sigma_p\rangle_{p<\omega}$ is a subsequence of $\langle s(n_1,n_2,\cdots,n_{2k-3},n_{2k-2})\rangle_{n_{2k-2}<\omega}$.

Similarly, if $\langle \sigma_p\rangle_{p<\omega}$ is a sequence in S converging to $(\langle r_1,r_2,\cdots,r_k\rangle,\langle s_1,s_2,\cdots,s_\ell\rangle)$ so that the σ_p^2's have length ℓ, then $\langle \sigma_p\rangle_{p<\omega}$ is eventually a subsequence of $\langle s(n_1,n_2,\cdots,n_{2\ell-2},n_{2\ell-1})\rangle_{n_{2\ell-1}<\omega}$.

Since $\langle s(n_1,n_2,\cdots,n_j,n_{j+1})\rangle_{n_{j+1}<\omega}$ converges to $s(n_1,n_2,\cdots,n_j)$ and these are essentially the only convergent sequences in S, S may be viewed as a closed copy of S_ω in σY^2.

C. *Example.* There is a regular space Z with a countable weak base [6] so that σZ^2 is not regular.

Let P be a countable set of irrationals which is dense in $\mathbf{I}$, where $\phi: P \to \mathbf{N}$ is one-to-one. Let Z be the set of all non-void sequences (finite or infinite) in P with wfc system defined as follows.

$B(n,\langle p_o,p_1,\cdots,p_k\rangle) = \langle p_o,p_1,\cdots,p_{k-1},S_n(p_k)\rangle \cup \langle p_o,p_1,\cdots,$
$p_k,S_n(0),\cdots\rangle,$
$B(n,\langle p_i\rangle_{i\in\omega}) = \langle p_o,p_1,\cdots,p_{n-1},p_n,\cdots\rangle$, where
$S_n(x) = \{y \in P: |y-x| < \frac{1}{n}\}$ and $\langle p_o,p_1,\cdots,p_k,T,\cdots\rangle$ is the set of all sequences in P, finite or infinite, which extend a member of $\langle p_o,p_1,\cdots,p_k,T\rangle$. Z is regular and has a countable weak base.

For $k \in \omega$ let

$$W_{2k} = \cup\{B(\phi(q_k),\langle p_o,p_1,\cdots,p_k\rangle) \times B(1,\langle q_o,q_1,\cdots,q_k\rangle): \quad p_o,p_1,\cdots,p_k,q_o,q_1,\cdots,q_k \in P\}$$

$$W_{2k+1} = \cup\{B(1,\langle p_o,p_1,\cdots,p_{k+1}\rangle) \times B(\phi(p_{k+1}),\langle q_o,q_1,\cdots,q_k\rangle): \quad p_o,p_1,\cdots,p_{k+1},q_o,q_1,\cdots,q_k \in P\}.$$

It is straightforward to check that a sequence converging to a member of W_k must eventually be in $W_k \cup W_{k+1}$, and hence $W = \cup_{k\in\omega}W_k$ is a sequentially open set in Z^2.

Let $\{p_o,q_o'\} \subset P$ so that $\phi(p_o) > (q_o')^{-1}$. We will show that every sequentially open set U in Z^2 with $(\langle p_o\rangle,\langle q_o'\rangle) \in U$ contains a sequence converging to a point not in W, hence that σZ^2 is not regular.

Assume we have found p_i $(i \leq k)$, q_i $(i < k)$, and q_k' so that

1. $(\langle p_o,\cdots,p_i\rangle,\langle q_o,\cdots,q_i\rangle) \in U$ if $i < k$.
2. $\phi(q_i) > (p_{i+1})^{-1}$ and $\phi(p_i) > q_i^{-1}$ if $i < k$.

3. $\phi(p_k) > (q_k')^{-1}$.

4. $(\langle p_0,\cdots,p_k\rangle,\langle q_0,\cdots,q_{k-1},q_k'\rangle) \in U$

Because of (4) we can find an $m \in \omega$ such that $B(m,\langle p_0,\cdots,p_k\rangle) \times B(m,\langle q_0,\cdots,q_{k-1},q_k'\rangle) \subset U$; choose $p_{k+1}' \in P$ so that $\langle p_0,\cdots,p_k,p_{k+1}'\rangle \in B(m,\langle p_0,\cdots,p_k\rangle)$ and $q_k \in P$ so that $\langle q_0,\cdots,q_k\rangle \in B(m,\langle q_0,\cdots,q_{k-1},q_k'\rangle)$, $q_k^{-1} < (q_k')^{-1} < \phi(p_k)$, and $\phi(q_k) > (p_{k+1}')^{-1}$. Note that $(\langle p_0,\cdots,p_k\rangle,\langle q_0,\cdots,q_k\rangle) \in U$.

Since $(\langle p_0,\cdots,p_k,p_{k+1}'\rangle,\langle q_0,\cdots,q_k\rangle) \in U$, there is an $n < \omega$ so that $B(n,\langle p_0,\cdots,p_k,p_{k+1}'\rangle) \times B(n,\langle q_0,\cdots,q_k\rangle) \subset U$. So there is a $q_{k+1}' \in P$ so that $\langle q_0,\cdots,q_k,q_{k+1}'\rangle \in B(n,\langle q_0,\cdots,q_k\rangle)$ and a $p_{k+1} \in P$ such that $\langle p_0,\cdots,p_{k+1}\rangle \in B(n,\langle p_0,\cdots,p_k,p_{k+1}'\rangle)$, $\phi(q_k) > (p_{k+1})^{-1}$, and $\phi(p_{k+1}) > (q_{k+1}')^{-1}$. This finishes the induction.

The sequence $\{(\langle p_0,\cdots,p_k\rangle,\langle q_0,\cdots,q_k\rangle): k \in \omega\}$ in U converges to $\zeta = (\langle p_i\rangle_{i<\omega},\langle q_i\rangle_{i<\omega})$ in Z^2. To see that $\zeta \notin W$, note that if $\zeta \in B(\phi(s_k), \langle r_0,r_1,\cdots,r_k\rangle) \times B(1,\langle s_0,s_1,\cdots,s_k\rangle)$ for some $k \geq 0$, then, since every infinite sequence in $B(\phi(s_k),\langle r_0,r_1,\cdots,r_k\rangle)$ is an extension of $\langle r_0,r_1,\cdots,r_k\rangle$, $\langle p_0,\cdots,p_k\rangle = \langle r_0,\cdots,r_k\rangle$, and for the same reason $\langle s_0,\cdots,s_k\rangle = \langle q_0,\cdots,q_k\rangle$. Thus $\zeta \in B(\phi(q_k),\langle p_0,p_1,\cdots,p_k\rangle) \times B(1,\langle q_0,\cdots,q_k\rangle)$, which would mean $\phi(q_k) < (p_{k+1})^{-1}$, violating (2). A like argument shows that ζ is not in any of the sets $B(1,\langle r_0,r_1,\cdots,r_{k+1}\rangle) \times B(\phi(r_{k+1}),\langle s_0,\cdots,s_k\rangle)$. Thus $\zeta \notin W$ as claimed.

We note that Z is an $\aleph_0$-space (Theorem 1.15 in [6]), thereby answering Michael's question in [4].

References

[1] A. Arhangel'skii and S. Franklin, *Ordinal invariants for topological spaces*, Mich. Math. J. 15 (1968), 313-320.

[2] S. Davis, G. Gruenhage, and P. Nyikos, *G_δ-sets in symmetrizable and related spaces*, General Topology and Appl. 9 (1978), 253-261.

[3] V. Kannan, *An extension that nowhere has the Fréchet property*, Mich. Math. J. 20 (1973), 225-234.

[4] E. Michael, *On* k-*spaces*, k_R-*spaces and* k(X), Pac. J. Math. 47 (1973), 487-498.

[5] P. Nyikos, *Metrizability and the Fréchet-Urysohn property in topological groups* (to appear).

[6] F. Siwiec, *On defining a space by a weak base*, Pac. J. Math. 52 (1974), 233-245.

University of Texas at El Paso

El Paso, Texas 79968