TOPOLOGY PROCEEDINGS
Volume 6, 1981
Pages 299–309
http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

The work of Ferry [3] and Torunczyk [7] proving that the homeomorphism group of a compact Q-manifold is an ℓ_2-manifold leads naturally to the search for characterizations of other homeomorphism groups. This paper deals with the spaces $R^\infty = \varinjlim R^n$ and $Q^\infty = \varinjlim Q^n$, where R denotes the reals and Q the Hilbert cube. Some topological properties of their homeomorphism groups are given in the author's doctoral dissertation. No complete characterization is now known.

The author would like to thank the referee for a careful reading of the paper and several helpful suggestions.

2. Preliminaries

Throughout this paper we use F to denote either R^∞ or Q^∞, and M will be a (paracompact) F-manifold. It is known [4, Theorem II-6] that F is homeomorphic to a topological vector space. We let $\mathcal{H}(X)$ be the group of homeomorphisms of a space X with the compact-open topology. Also, id_X denotes the identity map of X.

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[1]This research was done while the author was supported by a Susàn B. Riley Fellowship from the American Association of University Women, and constitutes part of the author's doctoral dissertation at Vanderbilt University.

If X is a space with a binary operation $\cdot$, and $A, B \subset X$, we write $A\cdot B = \{a\cdot b | a \in A, b \in B\}$. If X is a vector space over R, and if $t \in R$, $A \subset X$, we set $t\cdot A = \{t\cdot a | a \in A\}$.

Let G be a topological group with identity element e, and let X be a space. Recall that a *group action* α of G on X is a function $\alpha: G \times X \to X$ such that the induced map $\hat{\alpha}: G \to \mathcal{H}(X)$ is a homomorphism of groups. We do not require that α be continuous, but we do assume that each $\hat{\alpha}(g)$ is continuous.

We will find the following lemmas useful. Their proofs are routine and will be omitted.

Lemma A. *If* α *is continuous, then* $\hat{\alpha}$ *is continuous.*

Lemma B. *If* α *is a continuous group action of the topological group* G *on* X, *then the function* $\lambda: G \times \mathcal{H}(X) \to \mathcal{H}(X)$ *defined by*

$$\lambda(g,h) = \hat{\alpha}(g)\circ h$$

is continuous.

Since λ is just a restriction of the group operation, this result is trivial when $\mathcal{H}(X)$ is a topological group. In our case ($X = F$), this is not true.

3. The Theorem

Here we state the result which is the main tool used to generate the examples which follow in Section 4.

Theorem 1. *Let* α *be a continuous group action of a topological group* G *on a space* X. *Let* H *be a subset of*

$\mathcal{H}(X)$ containing $\hat{\alpha}(G)$ and satisfying

(i) $\hat{\alpha}(G) \circ H \subset H$.

Suppose there is a continuous map $r: H \to G$ *such that*

(ii) $r(id_X) = e$;

(iii) $r[\hat{\alpha}(g) \circ h] = g \cdot r(h)$, *for* $g \in G$, $h \in H$.

Then $H \cong G \times r^{-1}(e)$, *and* $\hat{\alpha}$ *is an embedding.*

Proof. Define $q: H \to r^{-1}(e)$ by

$$q(h) = \hat{\alpha}[r(h)^{-1}] \circ h.$$

We show that the desired homeomorphism $\phi: H \to G \times r^{-1}(e)$ is given by

$$\phi(h) = (r(h), q(h)).$$

First, $q(H) \subset H$ by (i), and (iii) implies that $rq(h) = e$. Thus, q is well-defined.

Define $\psi: G \times r^{-1}(e) \to H$ by

$$\psi(g,f) = \hat{\alpha}(g) \circ f.$$

The image of ψ is contained in H by (i).

Using property (iii) and the fact that $\hat{\alpha}$ is a homomorphism, it is easily shown that ϕ and ψ are inverses. Continuity of both maps follows from continuity of inversion in G and Lemma B.

Now, from properties (ii) and (iii) we obtain that $r\hat{\alpha} = id_G$, implying that $\hat{\alpha}$ is one-to-one and open onto its image. Lemma A gives continuity, and the proof is complete.

Remark. This theorem is a topological generalization of a standard result in abelian group theory. If H is a group with subgroup G, and $r: H \to G$ is a homomorphism fixed on G, then H is isomorphic to the direct sum of G and the kernel of r.

4. Applications

Example 1. *Let* G *be any topological group. Let* $\mathcal{H}_0(G) = \{h \in \mathcal{H}(G) | h(e) = e\}$. *Then* $\mathcal{H}(G) \cong G \times \mathcal{H}_0(G)$.

Proof. Define $\alpha: G \times G \to G$ by $\alpha(x,y) = x \cdot y$. It is easily verified that α is a continuous group action of G on itself. Define $r: \mathcal{H}(G) \to G$ by $r(h) = h(e)$. Conditions (i) and (ii) of the theorem are satisfied by $H = \mathcal{H}(G)$, and for (iii) we have

$$r[\hat{\alpha}(x) \circ h] = \hat{\alpha}(x)[h(e)] = \alpha(x,h(e)) = x \cdot h(e) = x \cdot r(h).$$

Observe that $r^{-1}(e) = \mathcal{H}_0(G)$ and the proof is complete.

In Example 1, the factors of $\mathcal{H}(G)$ are both groups. It is easily verified that the map r is not a group homomorphism, so the factorization is not an algebraic one.

Example 2. *Since* F *has a topological group structure*, $\mathcal{H}(F) \cong F \times \mathcal{H}_0(F)$, *by Example* 1.

Example 3. F *is a factor of* $\mathcal{H}(M)$.

Proof. Denote the group operation on F by +. Replace M by M × F [6, Theorem 1] and [5, Theorem 1] and fix $m_0 \in M$. Define $\alpha: F \times (M\times F) \to M\times F$ by $\alpha(x,(m,y)) = (m,x+y)$. Define $r: \mathcal{H}(M\times F) \to F$ by $r(h) = \pi_F h(m_0,0)$, where $0 \in F$ is the identity and π_F is projection. It is routine to verify that the theorem applies to α, r, and $H = \mathcal{H}(M\times F)$.

Corollary 3.1. $\mathcal{H}(M)$ *is* F-*stable; that is* $\mathcal{H}(M) \cong \mathcal{H}(M) \times F$.

Proof. Since $F \cong F \times F$, it suffices to show that F is a factor of $\mathcal{H}(M)$. This is Example 3.

Corollary 3.2. *$\mathcal{H}$(M) has the disjoint* n-*cube property* (see [8, Remark 3]).

Proof. It is not difficult to prove that if X has the disjoint n-cube property, then so does X × Y. By a general position argument and [8], F has this property. Apply Example 3.

Example 4. *Let* $\mathcal{H}_p(R^\infty) = \{h \in \mathcal{H}_0(R^\infty) | h(1,0,0,\cdots) = (t,0,0,\cdots)$, *for some* $t > 0\}$. *Then* $\mathcal{H}_0(R^\infty) \cong R^\infty \times \mathcal{H}_p(R^\infty)$.

Proof. Define a continuous norm on R^∞ by

$$|x| = |(x_1,x_2,\cdots)| = (\sum_{i=1}^{\infty} x_i^2)^{1/2}.$$

Since $x_i = 0$ for i sufficiently large, $|x|$ is well-defined, and agrees with the usual norm on the subspaces $R^n \subset R^\infty$. Let

$$S = \{x \in R^\infty | \ |x| = 1\}.$$

Then $S = \varinjlim S^n = S^\infty \cong R^\infty$ by [2, Corollary 4.3], where $S^n \subset R^{n+1}$ is the usual n-dimensional unit sphere. Thus, S has a group structure. We let $e = (1,0,0,\cdots)$ be the identity element and denote the operation by $*$.

Just as S^1 acts on R^2 by rotation, we can define a group action α: $S \times R^\infty \to R^\infty$ by

$$\alpha(x,y) = \begin{cases} |y|(x*\frac{y}{|y|}), & y \neq 0; \\ 0 & , y = 0. \end{cases}$$

We verify that α is indeed a group action. We have

$$\alpha(e,y) = \begin{cases} |y|(e*\frac{y}{|y|}), & y \neq 0; \\ 0 & , y = 0 \end{cases}$$

$= y$. Also, for $y \neq 0$, we have

$$\alpha(x*z,y) = |y|(x*z*\frac{y}{|y|}) = \alpha[x,|y|(z*\frac{y}{|y|})] = \alpha[x,\alpha(z,y)].$$

The same equality clearly holds for $y = 0$. Thus, α is a group action.

Now, $\hat{\alpha}(S) \subset \mathcal{H}_0(R^\infty)$. Since $\mathcal{H}_0(R^\infty)$ is a group, (i) of the theorem holds. Define $r\colon \mathcal{H}_0(R^\infty) \to S$ by

$$r(h) = \frac{h(e)}{|h(e)|}.$$

Notice that r is not defined on all of $\mathcal{H}(R^\infty)$. It is easy to show that r is continuous and satisfies (ii) and (iii) of the theorem. Also, $r^{-1}(e) = \mathcal{H}_p(R^\infty)$.

It remains to show continuity of the action α. By continuity of the operations involved, α is clearly continuous at all points (x,y) with $y \neq 0$. Further, $S \times R^\infty \cong R^\infty$ is a k-space. Thus we restrict our attention to a point $(x,0)$ contained in some compact $K \subset S \times R^\infty$.

We make the following observations:

(1) α is norm-preserving: that is, $|\alpha(x,y)| = |y|$, for all $y \in R^\infty$;

(2) α takes K into a compact set in R^∞.

To prove (2), we use the homeomorphism-isomorphism $R^\infty \to S$ to write $S = \varinjlim C_n$, where each C_n is compact in C_{n+1} and $C_n * C_n \subset C_{n+1}$. Then $\alpha(C_n \times t\cdot C_n) \subset t\cdot C_{n+1}$ by (1). By compactness, K is contained in a set of the form $C_i \times \cup\{tC_i | t \in [0,m]\}$ so (2) holds.

Now let W be a neighborhood of $\alpha(x,0) = 0$ in R^∞. By (2), $\alpha(K) \subset R^j$, for some $j \geq i$. Choose $\varepsilon > 0$ such that $A = \{y \in R^j \mid |y| < \varepsilon\} \subset W \cap R^j$. Let $V = K \cap (S \times A)$, a neighborhood of $(x,0)$ in K. Applying (1) it is easy to see that $\alpha(V) \subset W$.

Thus α is continuous and the proof is complete. Note that in this example the factor $\mathcal{H}_p(R^\infty)$ is not a subgroup of $\mathcal{H}_0(R^\infty)$.

Corollary 4.1. *$\mathcal{H}_0(R^\infty)$ is R^∞-stable.*

Corollary 4.2. $\mathcal{H}_0(R^\infty) \cong \mathcal{H}(R^\infty)$.

Proof. Apply Example 2 and 4.1.

We provide one further factorization of $\mathcal{H}(R^\infty)$.

Example 5. *Let* $\mathcal{H}_{0,1}(R^\infty) = \{h \in \mathcal{H}(R^\infty) \mid h(0) = 0$ *and* $h(e) = e\}$, e *as in Example* 4. *Then* $\mathcal{H}_p(R^\infty) \cong R \times \mathcal{H}_{0,1}(R^\infty)$.

Proof. Our topological model for R will be the multiplicative group $R_+ = (0,\infty)$. Define $\alpha\colon R_+ \times R^\infty \to R^\infty$ by $\alpha(t,x) = t\cdot x$. Then α is a continuous group action satisfying $\hat{\alpha}(R_+) \subset \mathcal{H}_p(R^\infty)$ and $\hat{\alpha}(R_+)\circ\mathcal{H}_p(R^\infty) \subset \mathcal{H}_p(R^\infty)$. Define $r\colon \mathcal{H}_p(R^\infty) \to R_+$ by $r(h) = h(e)$. (We are making the identification $(t,0,0,\cdots) = t \in R_+$.) Setting $H = \mathcal{H}_p(R^\infty)$, it is easily seen that the hypotheses of the theorem are satisfied. Since $r^{-1}(1) = \mathcal{H}_{0,1}(R^\infty)$, we are done.

Corollary 5.1. $\mathcal{H}(R^\infty) \cong R^\infty \times \mathcal{H}_{0,1}(R^\infty)$.

Proof. $\mathcal{H}(R^\infty) \cong \mathcal{H}_0(R^\infty)$ (4.2) $\cong R^\infty \times \mathcal{H}_p(R^\infty)$ (Example 4) $\cong R^\infty \times R \times \mathcal{H}_{0,1}(R^\infty)$ (Example 5) $\cong R^\infty \times \mathcal{H}_{0,1}(R^\infty)$.

We can show that $\mathcal{H}_{0,1}(R^\infty)$ is also R^∞-stable. Thus $\mathcal{H}_{0,1}(R^\infty) \cong \mathcal{H}(R^\infty)$. We may ask whether $\mathcal{H}(R^\infty)$ contains $\prod_1^\infty R^\infty$, or whether these two spaces are homeomorphic. For any subset C of R^∞, let $\mathcal{H}_C(R^\infty) = \{h \in \mathcal{H}(R^\infty) \mid h|_C = \mathrm{id}\}$.

When is $\mathcal{H}_C(R^\infty) \cong \mathcal{H}(R^\infty)$?

Example 6. Let $\mathcal{H}_n(R^\infty) = \{h \in \mathcal{H}(R^\infty) | h|_{R^n} = id\}$, $n \geq 1$. *Then* R^∞ *is a factor of* $\mathcal{H}_n(R^\infty)$.

Proof. Let $\pi_i: R^\infty \to R$ be projection onto the i^{th} component, and let $p: R^\infty \to R^\infty$ be defined by $p(x_1,x_2,\cdots) = (x_{n+1},x_{n+2},\cdots)$. Let α be the S-action of Example 4. Define β: $S \times R^\infty \to R^\infty$ by

$$\beta(x,y) = (y_1,\cdots,y_n,\pi_1\alpha(x,p(y)),\pi_2\alpha(x,p(y)),\cdots),$$

and define r: $\mathcal{H}_n(R^\infty) \to S$ by

$$r(h) = \frac{ph(e')}{|ph(e')|},$$

where $e' = (0,\cdots,0,1,0,0,\cdots)$, 1 in the $(n+1)^{st}$ place. Apply the theorem.

The last example is not a direct application of Theorem 1, but the technique is similar.

Example 7. Let $L(R^\infty)$ *be the set of linear homeomorphisms of* R^∞. *Then,* $L(R^\infty)$ *is* ℓ_2*-stable.*

Proof. Let $e_n = (0,\cdots,0,1,0,0,\cdots) \in R^\infty$, where 1 is in the n^{th} component. Then $\mathcal{B} = \{e_n | n \geq 1\}$ is a vector space basis for R^∞. Let s be the topological group $\prod_1^\infty (0,\infty)$ under coordinate-wise multiplication, which we denote by $\cdot$. Recall that $s \cong \ell_2$ [1].

Define a group action α: $s \times R^\infty \to R^\infty$ by

$$\alpha(t,x) = \alpha((t_1,t_2,\cdots),(x_1,x_2,\cdots)) = (t_1x_1,t_2x_2,\cdots).$$

Since the components of x are eventually zero, α is well-defined. It turns out that α is discontinuous. But $\alpha|_{s\times R^n}$: $s \times R^n \to R^n$ is continuous (it is essentially the dot product in R^n), and it follows easily that $\hat{\alpha}$ is continuous. Also we have that $\hat{\alpha}(s) \subset \mathcal{L}(R^\infty)$.

Define $r: \mathcal{L}(R^\infty) \to s$ by $r(h) = (|h(e_1)|, |h(e_2)|, \cdots)$. Then r is continuous and satisfies

(i) $r(id_{R^\infty}) = \underset{\sim}{1}$, where $\underset{\sim}{1} = (1,1,\cdots) \in s$;

(ii) $r[h\circ\hat{\alpha}(t)] = r(h)\cdot t$.

Compare (ii) to condition (iii) in Theorem 1. As in that theorem (i) and (ii) guarantee that $\hat{\alpha}$ is an embedding. Also as before, we may define $\phi: \mathcal{L}(R^\infty) \to s \times r^{-1}(\underset{\sim}{1})$ by

$$\phi(h) = (r(h), h\circ\hat{\alpha}[r(h)^{-1}]).$$

Now, ϕ^{-1} is computed similarly to Theorem 1, and continuity of these maps follows from the following analog of Lemma B.

Lemma 7.1. *For the action* α *defined above, the composition map* $\rho: \mathcal{L}(R^\infty) \times s \to \mathcal{L}(R^\infty)$ *given by* $\rho(h,t) = h\circ\hat{\alpha}(t)$ *is continuous.*

Proof. Let $\rho(h,t) = h\circ\hat{\alpha}(t) \in (K,W) \subset \mathcal{L}(R^\infty)$, where (K,W) is a typical subbasic neighborhood for the compact-open topology. Let n be such that $K \subset R^n$ [4], and choose a relatively compact neighborhood $\mathcal{O}$ of K in R^n such that $h\hat{\alpha}(t)[cl(\mathcal{O})] \subset W$. Now, $(cl(\mathcal{O}), h^{-1}(W))$ is a neighborhood of $\hat{\alpha}(t)$ in $\mathcal{L}(R^\infty)$, so there is a basic neighborhood $V \times \prod_{j+1}^{\infty}(0,\infty)$ of t in s, with V open in $\prod_{1}^{j}(0,\infty)$, and such that $V \times \prod_{j+1}^{\infty}(0,\infty)$ is contained in $\hat{\alpha}^{-1}[(cl(\mathcal{O}), h^{-1}(W))]$. We may assume that $j \geq n$. Write $t = (t_1,\cdots,t_j,\cdots)$, and choose a relatively compact neighborhood U of $(t_1,\cdots,t_j)$ in $\prod_{1}^{j}(0,\infty)$, with $cl(U) \subset V$.

As noted above, $\alpha|_{S\times R^n}$ is continuous. Hence, $C = \alpha[(cl(U) \times \{1\} \times \{1\} \times \cdots) \times cl(\mathcal{O})]$ is a compact subset of R^∞. We will argue that

(a) $(h,t) \in (C,W) \times (U \times \prod_{j+1}^{\infty} (0,\infty))$;

(b) $\rho[(C,W) \times (U \times \prod_{j+1}^{\infty} (0,\infty))] \subset (K,W)$.

It is clear that $t \in U \times \prod_{j+1}^{\infty} (0,\infty)$. Let $y = \alpha(u,x) \in C$, where $u = (u_1,\cdots,u_j,1,1,\cdots)$ and $x \in cl(\mathcal{O})$. Now $h(y) = h\alpha(u,x) = [h\hat{\alpha}(u)](x)$. But $u \in cl(U) \times \{1\} \times \cdots \subset V \times \prod_{j+1}^{\infty} (0,\infty)$, so $\hat{\alpha}(u) \in (cl(\mathcal{O}),h^{-1}(W))$. Since $x \in cl(\mathcal{O})$, $h[\hat{\alpha}(u)(x)] \in hh^{-1}(W) = W$, and (a) holds.

Now, let $(g,u) \in (C,W) \times (U \times \prod_{j+1}^{\infty} (0,\infty))$, and let $k \in K$. Then $[\rho(g,u)](k) = [g\hat{\alpha}(u)](k) = g\alpha(u,k)$. Now, $k = (k_1,\cdots,k_n,0,0,\cdots) \in cl(\mathcal{O}) \subset R^n$, and $u = (u_1,\cdots,u_n, \cdots,u_j,\cdots)$. We want to argue that $\alpha(u,k) \in C$. It need not be true that $u \in cl(U) \times \{1\} \times \cdots$, but if we define $u' = (u_1,\cdots,u_j,1,1,\cdots)$, then $u' \in cl(U) \times \{1\} \times \cdots$ and $\alpha(u,k) = (u_1k_1,\cdots,u_nk_n,0,0,\cdots) = \alpha(u',k)$ (since $j \geq n$), and this last element is in C. Thus (b) holds and the lemma is proved.

Remarks. Lemma 7.1 is necessary, since it is not hard to show that composition of functions in $\mathcal{L}(R^\infty)$ is, in general, discontinuous.

We were not able to obtain a generalization of Lemma 7.1 or of Example 7. The special nature of the action allowed this particular case. Note that even though r is defined on $\mathcal{H}_0(R^\infty)$, (ii) holds only for linear maps.

References

1. R. D. Anderson, *Hilbert space is homeomorphic to the countable infinite product of lines*, A.M.S. Bulletin 72 (1966), 515-519.
2. T. Dobrowolski, *Extension of Bessaga's negligibility technique to certain infinite-dimensional groups*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astr. Phys. 26 (1978), 535-545.
3. S. Ferry, *The homeomorphism group of a compact Hilbert cube manifold is an* ANR, Ann. Math. (2) 106 (1977), no. 1, 101-119.
4. R. E. Heisey, *Contracting spaces of maps on the countable direct limit of a space*, A.M.S. Transactions 193 (1974), 389-411.
5. ________, *Stability, open embeddings and classification of bounded weak-* manifolds*, Gen. Top. and Appl. 6 (1976), 327-338.
6. ________, *Stability, classification, open embeddings, and triangulation of* R^{∞}*-manifolds*, Proceedings, Int'l. Conf. on Geometric Topology, Polish Scientific Publishers, Warsawa, 1980, 193-196.
7. H. Torunczyk, *Absolute retracts as factors of normed linear spaces*, Fund. Math. 86 (1974), 53-67.
8. ________, *On* CE*-images of the Hilbert cube and characterization of* Q*-manifolds*, Fund. Math. 106 (1980), 31-40.

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