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by

Ernest P. Lane

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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INSERTION OF A CONTINUOUS FUNCTION AND X × I

Ernest P. Lane[1]

If g and f are real-valued functions defined on a topological space X such that $g < f$ (i.e., $g(x) < f(x)$ for each x in X), consider the problem of finding necessary and sufficient conditions in order for there to be a continuous function h defined on X such that $g < h < f$. Such conditions seem to have characteristics that keep them from being applicable in some cases. For example, the condition given in Theorem 3.11 of [2] requires that $-g$ must possess the same property as f and that the classes to which g and f belong are closed under sum, supremum, and infimum. The condition given in Theorem 2.1 of [5] places a restriction on $f - g$. This paper gives a necessary and sufficient condition that avoids these problems, but it is in terms of subsets of $X \times I$.

All functions considered are assumed to map into the interval $(0,1)$; I denotes the interval $[0,1]$. (In most situations there is no loss in generality in replacing R with $(0,1)$.) It is therefore convenient to let $C(X)$ represent the set of all continuous functions from X into $(0,1)$. The characterization that is given below is in terms of the following subsets of $X \times I$: If k maps X into $(0,1)$, let

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$$L_k = \{(x,t) \in X \times I: t \leq k(x)\}$$

and

$$U_k = \{(x,t) \in X \times I: t \geq k(x)\}.$$

It is observed in [6] that if k is continuous then L_k and U_k are zero sets in $X \times I$; this will be used below in a proof. It is convenient to use the following terminology. If k is a real-valued function then k is *z-lower semicontinuous* (zlsc) [respectively, *z-upper semicontinuous* (zusc)] if for each real number r, $\{x \in X: k(x) \leq r\}$ [respectively, $\{x \in X: k(x) \geq r\}$] is a zero set. (These functions have been used in [1] and in [8].)

Proposition 1. *If* g *and* f *are functions from* X *into* (0,1) *such that* $g < f$ *and such that* L_g *and* U_f *are completely separated in* $X \times I$, *then there is a continuous function* h *on* X *such that* $g < h < f$.

Proof. If g and f are functions from X into (0,1) such that $g < f$ and such that L_g and U_f are completely separated, let F be a continuous function from $X \times I$ into I such that if (x,t) is in L_g then $F(x,t) = 0$ and if (x,t) is in U_f then $F(x,t) = 1$. Define functions f_1 and g_1 from X into (0,1) as follows:

$$f_1(x) = \inf\{t \in I: (\{x\} \times [t,1]) \subset F^{-1}(\{1\})\}$$

and

$$g_1(x) = \sup\{t \in I: (\{x\} \times [0,t]) \subset F^{-1}(\{0\})\}.$$

Since $L_g \subset F^{-1}(\{0\})$ and $U_f \subset F^{-1}(\{1\})$, $g \leq g_1$ and $f_1 \leq f$. Since F is continuous it is easy to show that $g_1 < f_1$. Also, g_1 is zusc and f_1 is zlsc; the outline of an argument to show that f_1 is zlsc follows: If r is any real number in (0,1), define a function F_r from X into I by

$$F_r(x) = \inf\{F(x,t): r \leq t \leq 1\}.$$

Then F_r is a continuous function from X into I. Next observe that

$$\{x \in X: f_1(x) \leq r\} = \{x \in X: F_r(x) = 1\}:$$

For if $f_1(x) \leq r$ then $\{x\} \times [r,1] \subset F^{-1}(\{1\})$ and hence $F_r(x) = 1$. Conversely, if $F_r(x) = 1$ then $F(x,t) = 1$ for all t such that $r \leq t \leq 1$ and consequently $\{x\} \times [r,1] \subset F^{-1}(\{1\})$; thus $f_1(x) \leq r$. Since F_r is continuous, $\{x \in X: F_r(x) = 1\}$ is a zero set. Since $\{x \in X: f_1(x) \leq r\}$ is a zero set for any r in (0,1), it follows from the definition that f_1 is zlsc. A similar argument shows that g_1 is zusc. Thus $g \leq g_1 \leq f_1 \leq f$, g_1 is zusc and f_1 is zlsc; by Proposition 6.1 of [1], there is a continuous function h defined on X such that $g_1 < h < f_1$. Hence $g < h < f$ and this concludes the proof.

Remark. In the above proof the zero set $F^{-1}(\{1\})$ in $X \times I$ is used to define a zlsc function on X; in the following it is observed that a zlsc function on X corresponds to a zero set in $X \times I$. Tong [9] proves that a space X is perfectly normal if and only if each lower semicontinuous function on X is a pointwise limit of an increasing sequence of continuous functions. His proof is inductive and involves the complete separation of certain subsets of X. It is straightforward to observe that this proof yields the following result: If f is a zlsc function on any topological space X, there is an increasing sequence of continuous functions on X whose pointwise limit is f. (The converse of this is immediate from the definition of zlsc.) Consequently, if f is zlsc then U_f is a countable intersection of zero sets, and hence U_f is a zero set in $X \times I$.

The main result of the paper follows from Proposition 1.

Theorem 1. *Let* L(X) *and* U(X) *be classes of functions from* X *into* (0,1) *such that* $C(X) \subset L(X)$ *and* $C(X) \subset U(X)$. *The following are equivalent:*

(i) *For any* g *in* U(X) *and any* f *in* L(X) *such that* $g < f$ *there is an* h *in* C(X) *such that* $g < h < f$.

(ii) *For any* g *in* U(X) *and any* f *in* L(X) *such that* $g < f$ *then* L_g *and* U_f *are completely separated in* $X \times I$.

Proof. That (ii) implies (i) follows from Proposition 1 (and does not require the hypothesis that $C(X) \subset L(X)$ and $C(X) \subset U(X)$). Conversely, suppose that (i) is satisfied. If $g \in U(X)$, $f \in L(X)$ and $g < f$, then by (i) there exists h in C(X) such that $g < h < f$. Since $C(X) \subset L(X)$, apply (i) to g and h; there exists k in C(X) such that $g < k < h$. Since k and h are continuous L_k and U_h are zero sets; since $k < h$ it follows that L_k and U_h are disjoint. Since $L_g \subset L_k$ and $U_f \subset U_h$, the sets L_g and U_f are completely separated.

An application of this theorem is given below that involves normal semicontinuous functions. If f_* [respectively, f^*] denotes the lower [respectively, upper] limit function of f then f is normal lower semicontinuous (nlsc) in case $f = (f^*)_*$ and f is normal upper semicontinuous (nusc) in case $f = (f_*)^*$. Since any continuous function is nlsc and nusc, the following is immediate from Theorem 1.

Corollary 1. *Let* L(X) [*respectively,* U(X)] *denote the nlsc* [*respectively, nusc*] *functions from* X *into* (0,1). *The following are equivalent:*

(i) *For any* g *in* U(X) *and any* f *in* L(X) *such that* $g < f$ *there is an* h *in* C(X) *such that* $g < h < f$.

(ii) *For any* g *in* U(X) *and any* f *in* L(X) *such that* $g < f$ *then* L_g *and* U_f *are completely separated in* $X \times I$.

The proof of Theorem 3.2 of [6] shows that if X is any space such that g nusc, f nlsc and $g < f$ imply that L_g and U_f are completely separated in $X \times I$, then there is a continuous function h on X such that $g < h < f$. The proof of this uses the result that for g nusc and f nlsc then L_g and U_f are regular closed sets in $X \times I$. The general converse of Theorem 3.2 of [6] remains open: Does condition (i) of Corollary 1 (for g nusc and f nlsc) imply that arbitrary disjoint regular closed subsets of $X \times I$ are completely separated?

If $X \times I$ satisfies the equivalent conditions of Theorem 1 for various classes of functions, the corresponding characterization of the space X is known. If L(X) (respectively, U(X)) is the class of lower (respectively, upper) semicontinuous functions on X, the conditions of Theorem 1 are equivalent to X is normal and countably paracompact. (This follows from Theorem 4 of [3] or from Theorem 2 of [4].) If L(X) (respectively, U(X)) is the class of upper (respectively, lower) semicontinuous functions on X, then the conditions of Theorem 1 are equivalent to X is an extremally disconnected P-space that satisfies Baire's condition. (See Proposition 6.11 of [1].) For an extensive list of corresponding characterizations of X for various cases of the classes L(X) and U(X), see Theorem 4.2 of [7].

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Appalachian State University

Boone, NC 28608