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Cantor Group Actions

Earlier the author showed [L] that the pseudo-arc and certain other chainable continua admit periodic homeomorphisms for every period $n > 1$. A modification of this technique allows one to construct p-adic Cantor group actions on the pseudo-arc. Actually a somewhat stronger result is possible, involving Cantor group actions on n-od like continua.

A discussion of zero-dimensional compact transformation groups is given in [A2], where Anderson shows that every such group acts effectively on the Menger universal curve. In conversation with the author he has recently asked whether every Cantor group acts effectively on the pseudo-arc. The answer is not known.

We shall restrict our attention to Cantor groups of a particular form. Let k be a fixed positive integer, and for each positive integer i let $G_i = Z_{m(i,1)} \oplus Z_{m(i,2)} \oplus \cdots \oplus Z_{m(i,k)}$ be a finite abelian group. Suppose that for each i there is an epimorphism $f_i: G_{i+1} \to G_i$ such that $f_i = \Pi_{j=1}^{k} f_i^j$ where $f_i^j: Z_{m(i+1,j)} \to Z_{m(i,j)}$ is an epimorphism for each i,j. Let $h(i,j) = f_i^j(1_{ij})$, where 1_{ij} is the generator of $Z_{m(i+1,j)}$. Let $G = \varprojlim\{G_i, f_i\}$. With the above notation and hypotheses on G and the G_i's and f_i's we have the following.

Theorem. There exists a k-od *like continuum* C_k *such that* G *acts effectively on* C_k.

(The notation and epsilontics here are messy. For heuristic purposes, the author suggests the reader first look at the argument in [L]. What follows is a modification of that.)

Proof. First we construct a tree-like continuum on which G acts and then show that the continuum is k-od like. For any positive integer n, let T_n be the standard n-od, i.e. the continuum resulting from $[0,1] \times \{0,1,2,\cdots,n-1\}$ by identifying $\{0\} \times \{0,1,2,\cdots,n-1\}$.

Let $m(i) = \sum_{j=1}^{k} m(i,j)$ and $n(i,j) = \frac{m(i+1,j)}{m(i,j)}$. Let θ_i be a map from $T_{m(i+1)}$ onto $T_{m(i)}$ satisfying the following condition. The interval $[0,1] \times \{(\sum_{a=1}^{j-1} m(i+1,a)) + bn(i,j) + c\}$, where $c < n(i,j)$ and $bn(i,j) + c < m(i+1,j)$, is mapped linearly order preserving, onto $[0,1] \times \{(\sum_{d=1}^{j-1} m(i,d)) + b\}$.

We shall now construct a map $\Psi_i\colon T_{m(i+1)} \to T_{m(i+1)}$. The set $[0,1] \times \{\sum_{a=1}^{j-1} m(i+1,a), (\sum_{a=1}^{j-1} m(i+1,a)) + 1,\cdots, (\sum_{a=1}^{j} m(i+1,a)) - 1\}$ will be mapped onto itself and so we will restrict our attention to this set. For convenience of notation let $j_{i+1}(n) = (\sum_{a=1}^{j-1} m(i+1,a)) + n$, where $0 \leq n < m(i+1,j)$. Let $j_{i+1}^{\#}(bm(i,j) + c) = j_{i+1}(n(i,j) f_i^j(c) + b)$.

Let $\delta(i,j) = m(2,j) + 1$ and $N(i,j) = \Pi_{a=1}^{i} n(a,j)$. Inductively define $\epsilon(i+1,j) = (m(1,j) + 1)\Pi_{b=1}^{i} 2\delta(b,j)$, and $\delta_{(i+1,j)} = (2N(i+1,j)\sum_{c=1}^{\epsilon(i+1,j)-1} \frac{c}{\epsilon(i+1,j)}) + 1$. The map Ψ_i will be piecewise linear. On the intervals where it is linear it will magnify distances by a factor of $\delta(i,j)$.

The interval $[0,1] \times \{j_{i+1}(n)\}$ will be divided into subintervals, each of length an integral multiple of $\frac{1}{\epsilon(i+1,j)}$. Starting from $\{0\} \times \{j_{i+1}(n)\}$ there will first be $2n(i,j)$ intervals of length $\frac{1}{\epsilon(i+1,j)}$, then $2n(i,j)$ intervals of length $\frac{2}{\epsilon(i+1,j)}$, etc., and a final interval of length $\frac{1}{\delta(i,j)}$. The first interval will be mapped linearly, order preserving into $[0,1] \times \{j^{\#}_{i+1}(n)\}$, starting at $(0,j^{\#}_{i+1}(n))$; the next interval will be mapped linearly, order reversing, back to $(0,j^{\#}_{i+1}(n))$; the next interval linearly, order preserving into $[0,1] \times j^{\#}_{i+1}(n \hat{+} 1)$; the next interval linearly, order reversing into $[0,1] \times j^{\#}_{i+1}(n \hat{+} 1)$, etc. This is continued, going cyclically through arms as indicated by the permutation $j^{\#}_{i+1}$, with two consecutive intervals being mapped linearly out an arm and then back, each time multiplying lengths by a factor of $\delta(i,j)$. (Addition mod $m(i+1,j)$ is indicated by $\hat{+}$.)

G_i acts on $T_{m(i)}$ by permuting the arms of $T_{m(i)}$, where the permutation $j^{\#}_i$ corresponds to the action of $(0,0,0,\cdots, 1_{ij},\cdots,0)$. The actions of G_i and G_{i+1} on $T_{m(i)}$ and $T_{m(i+1)}$ respectively commute with f_i and $\Psi_i\Theta_i$. Thus $G = \varprojlim\{G_i,f_i\}$ acts effectively on $T = \varprojlim\{T_{m(i)},\Psi_i\Theta_i\}$. We need only show that T is k-od like.

To do this we construct a map $p_{i+1}: T_{m(i+1)} \to T_{m(i+1)}$ whose image is a k-od. If A is the pth interval (of those used in defining Ψ_i) of $[0,1] \times j^{\#}_{i+1}(n)$, map it linearly order preserving onto the $(p-n)$th interval of $[0,1] \times \{j^{\#}_{i+1}(0)\}$. (If $p-n$ is non-positive, A is mapped to $(0,j^{\#}_{i+1}(0))$.) Let $\Pi_i: T \to T_{m(i)}$ be the standard projection map, and $\hat{\Pi}_i = p_i\Pi_i$.

If $p_{i+1}(x) = p_{i+1}(y)$ then $\mathrm{dist}(\Psi_i(x),\Psi_i(y)) < 2m(i,j)\frac{\delta(i,j)}{\epsilon(i,j)} < 2m(1,j)$, for the appropriate j. If $r > i$ and $p_r(x) = p_r(y)$ then $\mathrm{dist}(\Psi_i\Theta_i\Psi_{i+1}\Theta_{i+1}\cdots\Psi_{r-1}\Theta_{r-1}(x), \Psi_i\Theta_i\Psi_{i+1}\Theta_{i+1}\cdots\Psi_{r-1}\Theta_{r-1}(y)) < 2^{1+i-r}m(1,j)$, for the appropriate j.

Thus if $\hat{\Pi}_r(x) = \hat{\Pi}_r(y)$ then $\mathrm{dist}(x,y) < \sum_{i=1}^{r-1} 2^{-i} 2^{1+i-r} m(1,j) + \sum_{i=r}^{\infty} 2^{1-i} = \sum_{i=1}^{r-1} 2^{1-r} m(1,j) + 2^{2-r}$. If for any $\epsilon > 0$ we choose r sufficiently large, then $\mathrm{diam}\ \hat{\Pi}_r^{-1}(x) < \epsilon$ for each x. Thus $T = C_k$ is k-od like.

If we had introduced crookedness into the $\Psi_i\Theta_i$'s, we could have made T the wedge of k pseudo-arcs.

Corollary 1. *G acts effectively on the wedge of* k *pseudo-arcs.*

If $k = 1$, G is p-adic, giving the following result.

Corollary 2. *Every* p-*adic group acts effectively on the pseudo-arc.*

Question. Does every Cantor group act effectively on the pseudo-arc?

There are non p-adic groups acting on other chainable continua, e.g. the wedge of two pseudo-arcs. It is not known if any such act on the pseudo-arc itself.

Since p-adic group actions are of interest on other spaces [S] and since many spaces have a continuous decomposition into pseudo-arcs, the following question has been suggested in conversations with Keesling.

Question. Under what conditions does a space $\underline{X}$ with a continuous decomposition into pseudo-arcs admit an effective p-adic Cantor group action which is an extension of the action on individual pseudo-arcs of the decomposition?

Homeomorphisms of Products with the Menger Curve

In [Br] Anderson showed that the space of homeomorphisms of the Menger universal curve M is totally disconnected. In [KKT] the Kuperbergs and Transue showed that if $\underline{X}$ is an arcwise connected, semi-locally simply connected continuum then every homeomorphism of $M \times \underline{X}$ preserves fibers of the form $\{m\} \times \underline{X}$. In [P1] Judy Phelps extended another result of the Kuperbergs and Transue to show that if h is a homeomorphism of $\Pi_{a \in A} M_a$, a product of Menger curves or Sierpinski curves, then $h = \Pi_{a \in A} h_{s(a)}$, where s is a permutation of A and $h_{s(a)}$ is a homeomorphism from M_a to $M_{s(a)}$. In [P2] she showed that any product of the Menger curve with a nondegenerate continuum $\underline{X}$ is never representable, and hence by a result of hers never 2-homogeneous.

However there are products of the Menger curve with other continua which are Galois. A space $\underline{X}$ is *Galois* [FS] if for each $x \in \underline{X}$ and open U containing x there exists a homeomorphism $h: \underline{X} \to \underline{X}$ with $h(x) \neq x$ and $h|_{\underline{X}-U} = id_{\underline{X}-U}$. $\underline{X}$ is *isotopy Galois* [DFM] if for each $x \in \underline{X}$ and open U containing x there is an isotopy $F: \underline{X} \times [0,1] \to \underline{X}$ with $F(y,0) = y$ for all $y \in \underline{X}$, $F(x,1) \neq x$, and $F(y,t) = y$ for all $y \notin U$. $\underline{X}$ is *representable* [F] if for each $x \in \underline{X}$ and open U containing x, there exists an open V with $x \in V \subset U$ such that for each $y \in V$ there is a homeomorphism $h: \underline{X} \to \underline{X}$

with $h(x) = y$ and $h|_{\underline{\overline{X}}-U} = id_{\underline{\overline{X}}-U}$.

Fletcher, Snider, Duvall, and McCoy [DFM], [F], [FS] have studied representable spaces and Galois spaces. In [LP] some observations were made about these and a few questions raised. One of the questions raised was whether there is a (homogeneous) continuum N such that $M \times N$ is Galois but not isotopy Galois. We have recently been able to provide a negative answer to this question. The lemmas below provide this answer as well as some other observations about maps of products with a Menger curve factor.

Lemma 1. *If* f *and* g *are two distinct maps of* M *into itself* (M = Menger universal curve), *with* f *a homeomorphism, then there is a separation* A, B *of* M^M (the space of maps of M into M) *with* $f \in A$ *and* $g \in B$.

Proof. Consider M as a subset of the standard three-dimensional cube, with projections onto Sierpinski curves on each face. Let $x \in M$ such that $f(x) \neq g(x)$ and let U be an open set containing x such that $\Pi_i \circ f(U) \cap \Pi_i \circ g(U) = \emptyset$, where Π_i is the projection of M onto one of its Sierpinski curve faces, and $\Pi_i \circ f(U)$ and $\Pi_i \circ g(U)$ are each contractible in the complement of the other in the plane containing the appropriate Sierpinski curve face of M.

In $\Pi_i \circ f(U)$ there is a "boundary curve" C of the Sierpinski curve and a loop L in M with $f(L) \subset f(U)$ and $\Pi_i \circ f\colon L \to C$ a homeomorphism. Let r be the retraction of the Sierpinski curve onto C, projecting radially in the plane containing it.

Consider the two sets in M^M:

$A = \{h \in M^M\colon r \circ \Pi_i \circ h(L)$ is homotopically non-trivial in $C\}$

and

$B = \{h \in M^M\colon r \circ \Pi_i \circ h(L)$ is homotopically trivial in $C\}$.

Each of A and B is open, $f \in A$, $g \in B$, they are disjoint, and their union is M^M. This is the desired separation.

A similar result holds for the Sierpinski curve itself. M^M does contain arcs, but no homeomorphism lies on such.

Lemma 2. *Let* $\underline{\overline{X}}$ *be a compact metric space and* $\underline{\overline{Y}}$ *be any topological space. Let* $f\colon \underline{\overline{X}} \times \underline{\overline{Y}} \to \underline{\overline{X}} \times \underline{\overline{Y}}$ *be continuous. For each* $y \in \underline{\overline{Y}}$, *let* $f_y\colon \underline{\overline{X}} \to \underline{\overline{X}}$ *be defined by* $f_y(x) = \Pi_{\underline{\overline{X}}} \circ f((x,y))$. *Then* $f_*\colon \underline{\overline{Y}} \to \underline{\overline{X}}^{\underline{\overline{X}}}$ (the space of maps of $\underline{\overline{X}}$ into $\underline{\overline{X}}$) *defined by* $f_*(y) = f_y$ *is continuous.*

Proof. Let $\epsilon > 0$ and $y \in \underline{\overline{Y}}$. Let $\mathcal{U}$ be an open cover of $\underline{\overline{X}}$ by sets of diameter less than ϵ. Let $\hat{\mathcal{U}} = \{f^{-1} \circ \Pi_{\underline{\overline{X}}}^{-1}(U) \mid U \in \mathcal{U}\}$. For each $x \in \underline{\overline{X}}$, let $\underline{\overline{V}}_x$ be a product neighborhood of (x,y) contained in some $f^{-1} \circ \Pi_{\underline{\overline{X}}}^{-1}(U)$. Since $\underline{\overline{X}}$ is compact, there is a minimal finite subcollection $\underline{\overline{V}}_{x_1}, \underline{\overline{V}}_{x_2}, \cdots, \underline{\overline{V}}_{x_n}$ of the $\underline{\overline{V}}_x$'s covering $\underline{\overline{X}} \times \{y\}$. Let $A = \{z \in \underline{\overline{Y}} \mid \underline{\overline{X}} \times \{z\} \subset \cup_{i=1}^n \underline{\overline{V}}_{x_i}\}$. A is an open set containing y (in fact the intersection of the $\underline{\overline{Y}}$-factors of the $\underline{\overline{V}}_{x_i}$'s). Suppose $z \in A$ and $x \in \underline{\overline{X}}$. Then (x,y) and (x,z) are in a common V_x and so in a common $f^{-1} \circ \Pi_{\underline{\overline{X}}}^{-1}(U)$. Thus $\Pi_{\underline{\overline{X}}} \circ f((x,z))$ and $\Pi_{\underline{\overline{X}}} \circ f((x,y))$ are both in U and hence within ϵ of each other. So $f_*(A) \subset N_\epsilon(f_y)$ and f_* is continuous.

Lemma 3. *Let* M *be either the Menger universal curve or the Sierpinski universal curve and* $\underline{\overline{X}}$ *be any connected topological space. If* f: $M \times \underline{\overline{X}} \to M \times \underline{\overline{X}}$ *is any map such that, for some* $x \in \underline{\overline{X}}$, $\Pi_M \circ f\colon M \times \{x\} \to M$ *is a homeomorphism, then for each* $m \in M$ *and* $a,b \in \underline{\overline{X}}$, Π_m of $((m,a)) = \Pi_m$ of $((m,b))$ (i.e. f preserves $\{m\} \times \underline{\overline{X}}$ fibers).

Proof. Define $f_*\colon \underline{\overline{X}} \to M^M$ as above. f_x is a homeomorphism and so by Lemma 1 is a component of M^M. $f_*(\underline{\overline{X}})$ is connected and contains f_x. Thus $f_*(\underline{\overline{X}}) = f_x$.

Lemma 4. *Let* M *be either the Menger universal curve or the Sierpinski universal curve, and let* $\underline{\overline{X}}$ *be any connected topological space such that each* $x \in \underline{\overline{X}}$ *is contained in a proper open subset of* $\underline{\overline{X}}$. *Then* $M \times X$ *is Galois iff both* $\underline{\overline{X}}$ *and* $M \times \underline{\overline{X}}$ *are isotopy Galois.*

Proof. Suppose $M \times \underline{\overline{X}}$ is Galois. Let $(m,x) \in \underline{\overline{X}}$ and let $U \times \underline{\overline{V}}$ be an open set containing (m,x) with $U \neq M$ and $V \neq \underline{\overline{X}}$. Let h: $M \times \underline{\overline{X}} \to M \times \underline{\overline{X}}$ be a homeomorphism with $h((m,x)) \neq (m,x)$ and $h|_{M\times\underline{\overline{X}}-U\times V} = id_{M\times\underline{\overline{X}}-U\times\underline{\overline{V}}}$. By lemma 3 h preserves $\underline{\overline{X}}$ coordinates. Let A be an arc from m to a point of M-U. Then $\Pi_{\underline{\overline{X}}} \circ h\colon A \times \underline{\overline{X}} \to \underline{\overline{X}}$ is an isotopy of $\underline{\overline{X}}$, fixed outside V, which moves x. Thus $\underline{\overline{X}}$ is isotopy Galois, and so $M \times \underline{\overline{X}}$ is also isotopy Galois.

The converse is obvious.

We also have the following slight strengthening of a result in [P2].

Corollary. *If* $\underline{\overline{X}}$ *satisfies the hypotheses in lemma* 4, *then* $M \times \underline{\overline{X}}$ *is not representable.*

A partial solution of the M × $\underline{\overline{X}}$ Galois result was pointed out to me by Judy Phelps.

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