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RETRACTION OF M_1-SPACES

Takuo Miwa

In this paper, we shall prove that an M_1-space X can be imbedded in an M_1-space Z(X) as a closed subset in such a way that X is an AR($\mathcal{M}_1$) (resp. ANR($\mathcal{M}_1$)) if and only if X is a retract (resp. neighborhood retract) of Z(X), where $\mathcal{M}_1$ is the class of all M_1-spaces. Moreover, we shall prove that an M_1-space is an AE($\mathcal{M}_1$) (resp. ANE($\mathcal{M}_1$)) if and only if it is an AR($\mathcal{M}_1$) (resp. ANR($\mathcal{M}_1$)).

1. Introduction

In metric spaces, the closed imbedding theorem of Eilenberg-Wojdyslawski plays an important role in the development of retract theory. By using this theorem, it was shown that a metric space is an AE($\mathcal{M}$) (resp. ANE($\mathcal{M}$)) if and only if it is an AR($\mathcal{M}$) (resp. ANR($\mathcal{M}$)), where $\mathcal{M}$ is the class of all metric spaces. In [3], R. Cauty showed that a stratifiable space X can be imbedded in a stratifiable space Z(X) as a closed subset in such a way that X is an AR($\mathcal{S}$) (resp. ANR($\mathcal{S}$)) if and only if X is a retract (resp. neighborhood retract) of Z(X), where $\mathcal{S}$ is the class of all stratifiable spaces. By using this theorem, R. Cauty extended to stratifiable spaces the results of O. Hanner [6] concerning near maps and small homotopies. In this paper, for a space X we shall construct Z(X) by using the method of R. Cauty [3], and prove the results mentioned above. Furthermore, we consider the relationships between near maps, small homotopies, connectivity and AR($\mathcal{M}_1$) (or ANR($\mathcal{M}_1$)).

Throughout this paper, all spaces are assumed to be Hausdorff topological spaces and all maps to be continuous. N and I denote the set of all natural numbers and the closed unit interval [0,1], respectively. For the definitions of M_1-space and stratifiable space, see [4]. AR($\mathcal{C}$) (resp. ANR($\mathcal{C}$)) is the abbreviation for absolute (resp. neighborhood) retract for the class $\mathcal{C}$ and AE($\mathcal{C}$) (resp. ANE($\mathcal{C}$)) the abbreviation for absolute (resp. neighborhood) extensor for the class $\mathcal{C}$. For these definitions, see [8]. Note that in [8] each class $\mathcal{C}$ is weakly hereditary; that is to say, if $\mathcal{C}$ contains X, then it contains every closed subspace of X. However, in this paper we consider the class $\mathcal{M}_1$ of all M_1-spaces though it is unknown if $\mathcal{M}_1$ is weakly hereditary.

2. Auxiliary Lemma

Definition 2.1 ([12]). Let X be a space and F a closed subset of X. An open cover of $X - F$ is said to be an *anti-cover* of F. An anti-cover $\mathcal{V}$ is said to be *uniformly approaching* to F in X if for each open subset U of X, $Cl_X(\mathcal{V}(X - U))$ does not meet $U \cap F$, where $\mathcal{V}(X - U)$ denotes the star of $X - U$ with respect to $\mathcal{V}$ and Cl_X denotes the closure operation in X. A paracompact σ-space X is said to be a D-*space* if each closed subset of X has a uniformly approaching anti-cover.

Note that $\mathcal{V}$ is a semi-canonical cover of a pair (X,F) ([9]) if and only if $\mathcal{V}$ is uniformly approaching to F in X.

The following lemma was essentially proved in the proof of [11, Lemma 3.2]. For extensions of a closure preserving open collection, see [13, Definition 2].

Lemma 2.2. *Let* X *be a* D*-space,* F *a closed subset of* X *and* f *a map from* F *into a space* Y. *Let* Y *also denote the natural imbedding of* Y *in* $X \cup_f Y = Z$. *If* $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ *is a closure preserving open collection in* Y, *then for each* $\alpha \in A$ *there is a collection* $\{U'_\beta : \beta \in B_\alpha\}$ *of open subsets in* Z *satisfying the following three conditions:*

(E1) $\mathcal{U}' = \{U'_\beta : \beta \in B_\alpha, \alpha \in A\}$ *is closure preserving in* Z,

(E2) *for each* $\beta \in B_\alpha$, $U'_\beta \cap Y = U_\alpha$, *and for every open subset* V *in* Z *with* $V \cap Y = U_\alpha$ *there is* $\beta \in B_\alpha$ *such that* $U_\alpha \subset U'_\beta \subset V$, *and*

(E3) *for every open subset* W *in* Y, *there is an open subset* W' *of* Z *such that* $W' \cap Y = W$ *and* $W' \cap U'_\beta = \emptyset$ *whenever* $\beta \in B_\alpha$ *and* $W \cap U_\alpha = \emptyset$.

Proof. Let p be the projection from the free union $X \cup Y$ to Z. Since X is a D-space, X is an M_1-space. Therefore X is monotonically normal. Let G be a monotone normality operator for X satisfying the properties in [7, Lemma 2.2]. Since X is a D-space, F has a uniformly approaching anti-cover $\mathcal{V} = \{V_\lambda : \lambda \in \Lambda\}$ in X. In particular, since X is hereditarily paracompact, we may assume that $\mathcal{V}$ is locally finite in $X - F$. For each $U_\alpha \in \mathcal{U}$, let $U'_\alpha = \cup\{G(x, F - p^{-1}(U_\alpha)) : x \in p^{-1}(U_\alpha)\}$. Then U'_α is obviously open in X. For each $\alpha \in A$, let $B_\alpha = \{\gamma(\alpha) \subset \Lambda : p^{-1}(U'_{\gamma(\alpha)})$ is open in $U'_\alpha\}$, where $U'_{\gamma(\alpha)} = U_\alpha \cup p(\cup\{V_\lambda : \lambda \in \gamma(\alpha)\})$. Let

$B = \cup\{B_\alpha: \alpha \in A\}$, and $\mathcal{U}' = \{U'_\beta: \beta \in B\}$. Then condition (E2) is obviously satisfied by $\mathcal{U}'$, because for each open subset V in Z with $V \cap Y = U_\alpha$ there is a set $U'_\beta = U_\alpha \cup p(\cup\{V_\lambda \in \mathcal{V}: V_\lambda \subset p^{-1}(V) \cap U'_\alpha\})$, for some $\beta \in B_\alpha$, such that $U_\alpha \subset U'_\beta \subset V$. To prove (E3), let W be an open subset in Y. Then it is easy to see that $W' = W \cup p(\cup\{G(x, F - p^{-1}(W)): x \in p^{-1}(W)\})$ is an open subset of Z satisfying (E3).

Finally, to prove (E1), let $x \notin Cl_Z U'_\beta$ for all $\beta \in B' \subset B$. Then we shall prove that $x \notin Cl_Z(\cup\{U'_\beta: \beta \in B'\})$. First, assume that $x \in Y$ and also that $A' = \{\alpha \in A: B_\alpha \cap B' \neq \emptyset\}$. Then $x \notin Cl_Y U_\alpha$ for $\alpha \in A'$. Since $\mathcal{U}$ is closure preserving in Y, x has a neighborhood W in Y such that $W \cap U_\alpha = \emptyset$ for $\alpha \in A'$. By condition (E3), there is a neighborhood W' of x in Z such that $W' \cap U'_\beta = \emptyset$ for $\beta \in B'$. This proves that $\mathcal{U}'$ is closure preserving at $x \in Y$. Next, let $x \in Z - Y$. Then since $\mathcal{V}$ is locally finite in $X - F$, it is easily verified that there is a neighborhood W of x such that $W \cap U'_\beta = \emptyset$, for each $\beta \in B'$. This proves that $\mathcal{U}'$ is closure preserving at $x \in Z - Y$. Thus (E1) is satisfied by $\mathcal{U}'$. This completes the proof.

3. Construction of Z(X)

Construction 3.1. Let X be a space. M(X) denotes the full simplicial complex which has all points of X as the set of vertices. Then there is a canonical bijection i from the 0-skeleton M^0 of M(X) onto X. Let $Z' = M(X) \cup_i X$ be the adjunction space and $p': M(X) \cup X \to Z'$ the projection. By the aid of p', we identify X with $p'(X) \subset Z'$.

Since the restriction of p' to $M(X)$ is a bijection from $M(X)$ onto Z', by the abuse of language, a simplex σ of $M(X)$ is said to be contained in a subset U of Z' if $p'(\sigma)$ is contained in U. $Z(X)$ denotes the space such that Z' is the underlying set of $Z(X)$ and the topology of $Z(X)$ has a base which consists of a collection of sets U, which is open in Z', satisfying the following condition:

(C) If σ is a simplex of $M(X)$ such that all vertices of σ are contained in $U \cap X$, then σ is contained in U.

Let $p: M(X) \cup X \to Z(X)$ be the projection. Then p is obviously continuous. Let M^n be the n-skeleton of $M(X)$ and $Z^n = p(M^n \cup X)$.

Lemma 3.2. *If* X *is an* M_1*-space, then* $Z(X)$ *is also* M_1.

Proof. For each $n \in N$, let Y be the free union of all $(n+1)$-simplexes of $M(X)$, F the boundary of Y and $f: F \to Z^n$ the map defined by $f(x) = p(x)$ for $x \in F$. Then the set $Y \cup_f Z^n$ is equal to the set Z^{n+1}. Let $\{U_\alpha: \alpha \in A\}$ be a closure preserving open collection in Z^n. Since Y is a metric space, Y is a D-space. Therefore the technique of proof of Lemma 2.2 yields that, for each $\alpha \in A$, there is a collection $\{U'_\beta: \beta \in B_\alpha\}$ of open subsets in Z^{n+1} satisfying (E1), (E2) and (E3). (Note that this proof is slightly different from that of Lemma 2.2; i.e. if σ is $(n+1)$-simplex and U_α contains all vertices of σ, then σ is contained in U'_β, $\beta \in B_\alpha$.)

Now, let $\{U(\alpha_1): \alpha_1 \in A\}$ be a closure preserving open collection in X $(= Z^0)$. From the preceding paragraph we get that every $U(\alpha_1)$ can be extended to open subsets

$\{U(\alpha_1,\alpha_2): \alpha_2 \in A(\alpha_1)\}$ in Z^1 in such a way that the collection $\{U(\alpha_1,\alpha_2): \alpha_1 \in A, \alpha_2 \in A(\alpha_1)\}$ satisfies (E1), (E2) and (E3). Similarly, every $U(\alpha_1,\alpha_2)$ can be extended to open subsets $\{U(\alpha_1,\alpha_2,\alpha_3): \alpha_3 \in A(\alpha_1,\alpha_2)\}$ in Z^2 in such a way that the collection $\{U(\alpha_1,\alpha_2,\alpha_3): \alpha_1 \in A, \alpha_2 \in A(\alpha_1),$ $\alpha_3 \in A(\alpha_1,\alpha_2)\}$ satisfies (E1), (E2) and (E3). Repeating this process, we get for each $n \in N$ a closure preserving open collection $\{U(\alpha_1,\cdots,\alpha_{n+1}): \alpha_1 \in A, \alpha_2 \in A(\alpha_1),\cdots,$ $\alpha_{n+1} \in A(\alpha_1,\cdots,\alpha_n)\}$ in Z^n. Let $\Sigma = \{(\alpha_1,\alpha_2,\alpha_3,\cdots):$ $\alpha_1 \in A, \alpha_2 \in A(\alpha_1), \alpha_3 \in A(\alpha_1,\alpha_2),\cdots\}$. For each $(\alpha_1,\alpha_2,\cdots) \in \Sigma$, let $U(\alpha_1,\alpha_2,\cdots) = \cup\{U(\alpha_1,\cdots,\alpha_n): n \in N\}$. Then $U(\alpha_1,\alpha_2,\cdots)$ is open in $Z(X)$, because, for each $n \in N$, $U(\alpha_1,\alpha_2,\cdots) \cap Z^n = U(\alpha_1,\cdots,\alpha_{n+1})$ is open in Z^n and $U(\alpha_1,\alpha_2,\cdots)$ satisfies (C) by the construction of $U(\alpha_1,\cdots,\alpha_n)$. Next, we claim that $\mathcal{U} = \{U(\alpha_1,\alpha_2,\cdots):$ $(\alpha_1,\alpha_2,\cdots) \in \Sigma\}$ is closure preserving in $Z(X)$. Let $x \in Z^0$ $(= X)$ and $x \notin Cl_{Z(X)}U(\alpha_1,\alpha_2,\cdots)$ for all $(\alpha_1,\alpha_2,\cdots) \in$ $\Sigma' \subset \Sigma$. Then $x \notin Cl_X U(\alpha_1)$ for all $\alpha_1 \in A' = \{\alpha_1:$ $(\alpha_1,\alpha_2,\cdots) \in \Sigma'\}$. Since $\{U(\alpha_1): \alpha_1 \in A'\}$ is closure preserving in X, x has an open neighborhood W_1 in X such that $W_1 \cap U(\alpha_1) = \emptyset$ for each $\alpha_1 \in A'$. Let W_2 be an open extension of W_1 to Z^1 which satisfies (E3). Namely, $W_2 \cap U(\alpha_1,\alpha_2) = \emptyset$ for all $\alpha_1 \in A'$ and $\alpha_2 \in A(\alpha_1)$. Repeating this process, we have for each $n \in N$ an open subset W_{n+1} in Z^n. Let $W = \cup\{W_n: n \in N\}$. Then W is an open neighborhood of x in $Z(X)$ such that $W \cap U(\alpha_1,\alpha_2,\cdots) = \emptyset$ for all $(\alpha_1,\alpha_2,\cdots) \in \Sigma'$. Thus $\mathcal{U}$ is closure preserving at $x \in Z^0$. This remains valid for $x \in Z^n$ with $n > 0$.

Finally, let $\{\mathcal{U}_n\}$ is a σ-closure preserving base for X. Then it is easily verified that the extensions $\{\mathcal{U}_n'\}$ of $\{\mathcal{U}_n\}$ to Z(X), by the same method above, is a σ-closure preserving base at each point of X. Furthermore, since M(X) is an M_1-space by [4, Theorem 8.3] and the open subspace Z(X) - X is homeomorphic to an open subspace of M(X), there exists a σ-closure preserving base $\{\mathcal{V}_n\}$ at each point of Z(X) - X. Thus $\{\mathcal{U}_n'\} \cup \{\mathcal{V}_n\}$ is a σ-closure preserving base for Z(X). This completes the proof.

Remark 3.3. It was shown in [3] that, if X is stratifiable, Z(X) is also stratifiable. If X is normal (resp. paracompact), Z' in Construction 3.1 is normal (resp. paracompact). By using this fact, it is easy to see that Z(X) is normal (resp. paracompact).

The following lemma was proved in [3, Lemma 1.2].

Lemma 3.4. *Let* X *be a space. If* Y *is a stratifiable space,* A *a closed subset of* Y *and* $f: A \to X$ *a map, then there is a map* $F: Y \to Z(X)$ *with* $F|A = f$.

The following theorem is an immediate consequence of Lemma 3.2 and 3.4.

Theorem 3.5. *An* M_1*-space* X *is an* $AR(\mathcal{M}_1)$ *(resp.* $ANR(\mathcal{M}_1)$*) if and only if* X *is a retract (resp. neighborhood retract) of* Z(X).

The following theorem is a direct consequence of Theorem 3.5 and Lemma 3.4. Note that whether the class $\mathcal{M}_1$

is weakly hereditary is a long-standing unsolved question first posed by Ceder [4].

Theorem 3.6. *An* M_1*-space is an* $AE(\mathcal{M}_1)$ *(resp.* $ANE(\mathcal{M}_1)$*) if and only if it is an* $AR(\mathcal{M}_1)$ *(resp.* $ANR(\mathcal{M}_1)$*).*

4. Near Maps, Small Homotopies and Connectivity

Definition 4.1 ([5]). A space Y is *equiconnected* if there is a map $F: Y \times Y \times I \to Y$ such that $F(x,y,0) = x$, $F(x,y,1) = y$ and $F(x,x,t) = x$ for all $(x,y) \in Y \times Y$ and $t \in I$. The space Y is said to be *locally equiconnected* if F is defined only on $U \times I$, for some neighborhood U of the diagonal of $Y \times Y$.

Definition 4.2 ([6]). Let $f, g: Y \to X$ be two maps. If X is covered by $\mathcal{U} = \{U_\alpha\}$, f and g are called $\mathcal{U}$*-near* if for each $y \in Y$ there is a $U_\alpha \in \mathcal{U}$ such that $f(y) \in U_\alpha$, $g(y) \in U_\alpha$.

Definition 4.3 ([6]). Let $h_t: Y \to X$ be a homotopy. If X is covered by $\mathcal{U} = \{U_\alpha\}$, h_t is called a $\mathcal{U}$*-homotopy* if for each $y \in Y$ there is a $U_\alpha \in \mathcal{U}$ such that $h_t(y) \in U_\alpha$ for all $t \in I$. The space Y is said to *dominate* the space X if there are two maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ is homotopic to the identity map of X. If the homotopy is a $\mathcal{U}$-homotopy for a covering $\mathcal{U}$ of X, Y is said to $\mathcal{U}$*-dominate* X.

Proposition 4.4. *If an* M_1*-space* Y *is an* $ANR(\mathcal{M}_1)$, *then* Y *is locally equiconnected.*

Proof. Let $A = Y \times Y \times \{0,1\} \cup \Delta \times I$, where Δ is the diagonal of $Y \times Y$. We define a function $f: A \to Y$ as follows: $f(x,y,0) = x$, $f(x,y,1) = y$ and $f(x,x,t) = x$ for $t \in I$. Then f is continuous. Since Y is an ANR($\mathcal{M}_1$), by Theorem 3.6 there is a neighborhood U of Δ in $Y \times Y$ and a map $F: U \times I \to Y$ such that $F|A = f$. Therefore Y is locally equiconnected.

Proposition 4.5. *Let an* M_1*-space* Y *be an* ANR($\mathcal{M}_1$). *For any open covering* $\mathcal{U}$ *of* Y, *there is an open covering* $\mathcal{V}$ *of* Y, *which is a refinement of* $\mathcal{U}$, *such that for any space* X *any two* $\mathcal{V}$*-near maps* $f,g: X \to Y$ *are* $\mathcal{U}$*-homotopic by a homotopy which is constant on the set* $\{x \in X: f(x) = g(x)\}$.

Proof. Since Y is locally equiconnected by Proposition 4.4, there are a neighborhood U of the diagonal of $Y \times Y$ and a map $F: U \times I \to Y$ such that $F(x,y,0) = x$, $F(x,y,1) = y$ and $F(x,x,t) = x$ for all $(x,y) \in U$ and $t \in I$. For any $y \in Y$, there is a neighborhood V_y of y such that $V_y \times V_y \subset U$ and $F(V_y \times V_y \times I) \subset U_\alpha$ for some $U_\alpha \in \mathcal{U}$. Let $\mathcal{V} = \{V_y: y \in Y\}$. Then if two maps $f,g: X \to Y$ are $\mathcal{V}$-near, we can define a map $h: X \times I \to Y$ by $h(x,t) = F(f(x),g(x),t)$. By this homotopy, it is easy to see that f and g are $\mathcal{U}$-homotopic, and if $f(x) = g(x)$, then $h(x,t) = f(x)$ for all $t \in I$. This completes the proof.

The following theorem 4.6, 4.7 and 4.8 can be proved by the methods used in the proofs of Theorem 1.5, 1.6 and 1.8 of [3], respectively. For the definition of $\mathcal{U}$-fine, see [3] p. 136 "petite d'ordre $\mathcal{U}$." For the definition of (locally) hyperconnected, see [10] or [1].

Theorem 4.6. *Let an* M_1*-space* X *be an* ANR($\mathcal{M}_1$). *For any open covering* $\mathcal{U}$ *of* X, *there is a simplicial complex with the Whitehead topology which* $\mathcal{U}$*-dominate* X.

Theorem 4.7. *Let an* M_1*-space* X *be an* ANR($\mathcal{M}_1$). *For any open covering* $\mathcal{U}$ *of* X, *there is an open covering* $\mathcal{V}$ *of* X *such that, if* L *is a subcomplex of a simplicial complex* K *and contains all vertices of* K, *then every* $\mathcal{V}$*-fine map from* L *into* X *is extended to a* $\mathcal{U}$*-fine map from* K *into* X.

Theorem 4.8. *An* M_1*-space is an* AR($\mathcal{M}_1$) *(resp.* ANR($\mathcal{M}_1$)*) if and only if it is (resp. locally) hyperconnected.*

Corollary 4.9. *If an* M_1*-space* Y *is an* AR($\mathcal{M}_1$), *for any space* X *the function space* X^Y *with the pointwise convergence topology is an* AE($\mathcal{S}$).

This corollary is proved by Theorem 4.8 [2, Theorem 2.2] and [1, Theorem 4.1].

Added in proof. Some results of Section 3 have been announced in *Retraction and extension of mappings of* M_1*-spaces*, Proc. Japan Acad. 58 (1982).

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