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by

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Let (X,d) be a metric space. A *Lebesgue number* for an open cover $\mathcal{U}$ of X is an $\varepsilon > 0$ such that for each point $p \in X$, the open ball $B_d(p,\varepsilon) = \{x \in X: d(p,x) < \varepsilon\}$ is contained in at least one member of $\mathcal{U}$. A *Lebesgue space* (*$\mathcal{L}$-space*) is a metric space such that every open cover of the space has a Lebesgue number. It is known that the $\mathcal{L}$-spaces are precisely those metric spaces for which every continuous real-valued function is uniformly continuous ([6, p. 112], [1, p. 12]). In particular, every compact metric space is an $\mathcal{L}$-space. We will show that there are $\mathcal{L}$-spaces which are not even *locally* compact. Furthermore, in Theorem 1 we characterize those metric spaces which contain nonlocally compact $\mathcal{L}$-subspaces. The proof of Theorem 1 shows how to construct such subspaces in the most general possible setting. In Theorem 2 we show that any $\mathcal{L}$-space must be locally compact "at most points." In Theorem 3 we obtain a simple necessary and sufficient condition in order that a metric space be a locally compact $\mathcal{L}$-space. This condition determines the structure of all such spaces. We will briefly discuss Theorem 3 in relation to results in [4] and [5]--see the Remark at the end of the paper.

Recall that a metric space (X,d) is said to be *totally bounded* provided that for each $\varepsilon > 0$, there exist finitely many points $p_1,p_2,\cdots,p_n$ of X such that $X = \cup\{B_d(p_i,\varepsilon): i = 1,2,\cdots,n\}$ [2, p. 22]. A subset A of a metric space

(X,d) is said to be *uniformly isolated* provided that there exists a $\delta > 0$ such that $d(x,x') \geq \delta$ for all $x,x' \in A$ such that $x \neq x'$ [5, p. 153]. Clearly, a metric space fails to be totally bounded if and only if it contains an infinite uniformly isolated subset.

For use later on, let us note the following lemma. It follows easily from Theorem 1 of [1].

Lemma 1. *A metric space* (X,d) *is an* $\mathcal{L}$*-space if and only if the set* L *of all limit points of* X *is compact and for each open subset* U *of* X *such that* $U \supset L$, $X - U$ *is uniformly isolated.*

The following theorem is our characterization of those metric spaces which contain nonlocally compact $\mathcal{L}$-subspaces.

Theorem 1. *Let* (M,D) *be a metric space. Then: Every* $\mathcal{L}$*-subspace of* M *is locally compact if and only if every point of* M *has a totally bounded neighborhood,* i.e., *if and only if the completion of* M *is locally compact.*

Proof. Assume that there is a point $p \in M$ such that no neighborhood of p is totally bounded. Then, as noted above, each neighborhood of p contains an infinite uniformly isolated subset. Let X_1 be an infinite uniformly isolated subset of the ball $B_D(p,2^{-1})$ such that $p \notin X_1$. Assume inductively that we have chosen infinite subsets X_i of $B_D(p,2^{-i}) - \{p\}$ for each $i = 1,2,\cdots,n$ $(n < \infty)$ such that $\cup_{i=1}^n X_i$ is uniformly isolated. Let

$$\delta = \inf\{d(p,x) : x \in \cup_{i=1}^n X_i\}$$

and note that $\delta > 0$. Let $\varepsilon = \min\{\delta/2, 2^{-n-1}\}$. Since $\varepsilon > 0$, there is an infinite uniformly isolated subset X_{n+1} of $B_D(p,\varepsilon)$ such that $p \notin X_{n+1}$. Note that $\cup_{i=1}^{n+1} X_i$ is uniformly isolated. Thus, we have inductively defined X_n for each $n = 1,2,\cdots$ such that, for each $k = 1,2,\cdots$, $\cup_{n=1}^{k} X_n$ is uniformly isolated. Let

$$X = \{p\} \cup (\cup_{n=1}^{\infty} X_n)$$

and let d denote the subspace metric for X obtained from D. We see that p is the only limit point of X and that if U is any open subset of X such that $p \in U$, then $X - U \subset \cup_{n=1}^{k} X_n$ for some $k < \infty$ and, thus, $X - U$ is uniformly isolated. Hence, by Lemma 1, (X,d) is an $\mathcal{L}$-space. Furthermore, as is easy to see, (X,d) is not locally compact at p. Therefore, we have proved half of Theorem 1. To prove the other half, assume that every point of M has a totally bounded neighborhood and let Y be an $\mathcal{L}$-subspace of M. It follows easily from Lemma 1 that Y is complete (the fact that $\mathcal{L}$-spaces are complete is also noted in [6, §8, p. 112]). Let $y \in Y$. From our assumption about M, there is a totally bounded neighborhood N(y) of y. We assume without loss of generality that N(y) is a closed subset of M. Then, $N(y) \cap Y$ is a closed neighborhood of y in Y. Since Y is complete, $N(y) \cap Y$ is complete. Thus, since $N(y) \cap Y$ is totally bounded, $N(y) \cap Y$ is compact [2, p. 22]. Hence, we have proved that each point of Y has a compact neighborhood in Y. Therefore, Y is locally compact. This completes the proof of Theorem 1.

For metric linear topological vector spaces, Theorem 1 becomes the following result:

Corlllary. A metric linear topological vector space is finite dimensional if and only if every subset which is an $\mathcal{L}$-space is locally compact.

Proof. The corollary follows immediately from Theorem 1 above and 7.8 of [3, p. 62].

The $\mathcal{L}$-spaces constructed in the proof of Theorem 1 fail to be locally compact at only one point. The question arises as to whether there is an $\mathcal{L}$-space which fails to be locally compact at every point of some nonempty open subset. The following result answers this question by showing that no such $\mathcal{L}$-space exists.

Theorem 2. *If* (X,d) *is an $\mathcal{L}$-space, then* X *is locally compact at each point of a dense open subset of* X.

Proof. Let $L = \{x: x \text{ is a limit point of } X\}$ and let $W = \{x \in X: X \text{ is locally compact at } x\}$. Note that $X - W \subset L$. Thus, since L is compact (by Lemma 1), $X - W$ can not contain any nonempty open subset of X. Hence, W is a dense subset of X. Clearly, W is an open subset of X. This completes the proof of Theorem 2.

We have the following characterization of locally compact $\mathcal{L}$-spaces:

Theorem 3. *A metric space* (X,d) *is a locally compact $\mathcal{L}$-space if and only if* X *is the union of a compact subspace and a uniformly isolated subspace.*

Proof. Assume that (X,d) is a locally compact $\mathcal{L}$-space. Let L denote the set of all limit points of X. By Lemma 1,

L is compact. Thus, since X is locally compact, L can be covered by finitely many open subsets of X whose closures are compact. Hence, there is an open subset U of X such that $L \subset U$ and such that the closure $\overline{U}$ of U is compact. By Lemma 1, $X - U$ is uniformly isolated. Therefore, writing $X = \overline{U} \cup (X - U)$, we see that we have proved half of Theorem 3. Now, to prove the other half of Theorem 3, assume that (X,d) is a metric space such that X is the union of a compact subspace C and a uniformly isolated subspace E. We first show that X is locally compact. Note that $E - C$ is a uniformly isolated open subspace of X. Thus, for each point $x \in E - C$, we see that $\{x\}$ is an open subset of X. Hence, clearly, X is locally compact at each point of $E - C$. To show that X is locally compact at each point of C, it suffices (since C is compact) to show that C is an open subset of X. Suppose that C is not an open subset of X. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ of points of $X - C$ such that $\{x_n\}_{n=1}^{\infty}$ converges to a point of C. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence of points of E, we have a contradiction to the assumption that E is uniformly isolated. Thus, C is an open subspace of X. This completes the proof that X is locally compact. Next we show using Lemma 1 that (X,d) is an $\mathcal{L}$-space. Let L denote the set of all limit points of X. It was shown above that for each $x \in E - C$, $\{x\}$ is an open subset of X. Hence, $L \supset C$. Thus, since L is a closed subset of X and since C is compact, we have that L is compact. Let U be an open subset of X such that $U \supset L$. We wish to show that $X - U$ is uniformly isolated (see Lemma 1). Note that

$$(*)\ X - U = (C - U) \cup (E - U)$$

Since $C - U$ is a closed subset of the compact set C, $C - U$ is compact. Since $U \supset L$, $C - U$ contains no limit point of X. Thus, $C - U$ must be finite. Since $E - U$ is a subset of the uniformly isolated space E, $E - U$ is uniformly isolated. It now follows from (*) that $X - U$ is uniformly isolated. Therefore, we can now use Lemma 1 to conclude that (X,d) is an $\mathcal{L}$-space. This completes the proof of Theorem 3.

Remark. Let E be a subset of a metric space (M,d). Consider the following two conditions on E:

(1) every continuous real-valued function on E is uniformly continuous;

(2) $E = E_1 \cup E_2$ where E_1 is compact and E_2 is uniformly isolated.

Recall from the beginning of this paper that (1) is equivalent to E being an $\mathcal{L}$-space. Assume that (1) holds and that (M,d) is locally compact. Since E is complete [5, Thm. 1], E is a closed, therefore locally compact subspace of M. Hence, by Theorem 3 above, (2) holds. With no restriction on (M,d), we see from Theorem 3 that (2) implies (1). In Theorem 4 of [5] it is shown that (1) implies (2) if closed and bounded subsets of (M,d) are compact. However, as we have just shown, (1) is equivalent to (2) in more general spaces (M,d). Similar generalizations of some other results in [5] are also possible to obtain by using our results. In connection with this we note that condition (2), which occurs frequently in results in [4] and [5], is really equivalent to E being a locally compact $\mathcal{L}$-space (by our Theorem 3).

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