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1. Introduction

This paper begins what the author hopes is a very thorough study of the nature of groups of homeomorphisms of homogeneous continua, since it seems that understanding these groups is crucial to determining exactly what homogeneity properties a continuum has, and what effect the homogeneity properties have on other properties.

It has been known for some time that the group of homeomorphisms of a compact metric space is a separable complete metric topological group. With Gerald Ungar's application of what has come to be known as Effros's theorem [E] to solve several old problems in homogeneity [U1, U2], the study of homogeneity was revitalized. Since then, a number of authors have used this theorem: see for example [A], [R]. Efros's theorem is a valuable tool.

What Gerald Ungar first noticed and made use of was that if X is a homogeneous continuum and H(X) is its space of homeomorphisms, (H(X),X) is a polish topological transformation group, and H(X) is transitive on X; so that by Effros' theorem for each $x \in X$, the map T_x: $H(X) \to X$ defined by $T_x(h) = h(x)$ is open and onto. (For definitions and theorem, see next section.) Now locally compact topological groups have been widely studied, as have topological transformation groups: the problem is that in nearly all the literature the groups studied have been locally compact. Unfortunately, at least in this aspect, it is the case that if X is a homogeneous continuum, H(X) is *not* locally compact, as we shall see later. Also these groups very much fail to be abelian.

2. Definitions, Notation, Background Theorems

In this paper a *continuum* is a compact, connected, metric space. A topological space X is *homogeneous* means that if x and y are points of X, then there is a homeomorphism h from X onto itself such that h(x) = y. N denotes the positive integers. If A is a collection of sets, A* denotes the union of the members of A.

If X is a topological space, H(X) denotes the set of all homeomorphisms from X onto itself. If X is a continuum, then H(X) is a complete separable metric topological group. The metric that we will use is the familiar "sup" metric; i.e., if d is a metric on X (compatible with its topology), and h and f are in H(X), then $\rho_d(h,f) = \text{lub}\{d(h(x),f(x)) \mid x \in X\}$. When no confusion arises, ρ_d will just be ρ. The sup metric (ρ) induces on H(X) the compact-open topology.

To say that (G,X) (where G is a topological group and X is a topological space) is a *topological transformation group* means (1) there is a continuous map $\phi: G \times X \to X$ such that if h,g are in G, x is in X, and 1 is the identity in G, then $\phi(gh,x) = \phi(g,\phi(h,x))$ and $\phi(1,x) = x$. A transformation group is *polish* if both G and X are polish, that is, they are both separable and metrizable by a complete metric. If $x \in X$, we will use G_x to denote the *stabilizer* *subgroup of* G *with respect to* x, i.e., $G_x = \{g \in G | g(x) = x\}$, and G(x) will denote the *orbit under* G *of* x in X, i.e., $G(x) = \{y \in X |$ there is some h in G such that $h(x) = y\}$. If $A \subseteq G$, $x \in X$, $A(x) = \{y \in X |$ there is some h in A such that $h(x) = y\}$. We will say that (G,X) is *transitive*, or that G is transitive on X, if whenever x and y are points of X, there is some h in G such that $\phi(h,x) = y$.

The following is a somewhat simplified statement of Effros's theorem [U2]:

Effros's Theorem. Suppose (G,X) *is a polish topological transformation group. Then the following are equivalent:*

(1) *For each* x *in* X, *the map* ϕ_x: $G/G_x \to G(x)$ *defined by* $\phi_x(gG_x) = g(x)$ *(where* $g \in G$*) is a homeomorphism from* G/G_x *onto* G(x).

(2) *Each orbit is a* G_δ *set in* X.

(3) *Each orbit is second category in itself.*

G. Ungar observed in [U2] that the following is equivalent to each of the 3 statements above in the theorem (with, of course, the same hypothesis): For each x in X, the map T_x: $G \to G(x)$ defined by $T_x(g) = g(x)$ is an open map of G onto G(x). Note that if X is a homogeneous continuum then $H(X)(x) = X$ for $x \in X$.

A couple of years ago, F. D. Ancel [A] proved a slightly different version of Effros's theorem, which is at times applicable where the original version was not (and vice versa). This version follows.

Ancel's Version of Effros's Theorem. Suppose the complete separable metric topological group G *acts transitively on a metric space* X. *Then* G *acts micro-transitively on* X *if and only if* X *has a complete metric.* (The action of G on X is *micro-transitive* if for every x in X and every neighborhood u of 1 in G, u(x) is a neighborhood of x in X.)

If $n \in N$, $F^n(x)$ denotes the *nth configuration space* of X, that is, $F^n(X) = \{(x_1,x_2,\cdots,x_n) | x_i \in X$ for each $i \leq n$, and $x_i = x_j$ iff $i = j\}$. The space X is n-*homogeneous* (*strongly* n-*homogeneous*) means that if $\{x_1,x_2,\cdots,x_n\}$ and $\{y_1,y_2,\cdots,y_n\}$ are both n-element subsets of X, then there is an h in H(X) such that $h\{x_1,x_2,\cdots,x_n\} = \{y_1,y_2,\cdots,y_n\}$ $(h(x_1,x_2,\cdots,x_n) = (y_1,y_2,\cdots,y_n))$. Ungar [U2] has shown that for a homogeneous continuum which is not the circle, n-homogeneity and strong n-homogeneity are equivalent.

A topological space is *Galois* if for each x in X and open u containing x there exists a homeomorphism h in H(X) with $h(x) \neq x$ and $h \restriction (X-u) = id_{X-u}$. X is an *isotopy Galois* space if for each x in X and open u containing x, there exists an isotopy $F: X \times [0,1] \to X$ with $F_0 = id_X$ (i.e., 1), $F_1(x) \neq x$, and $F_t \restriction (X-u) = id_{(X-u)}$ for each t in [0,1]. X is *homeotopically homogeneous* if for every x and y in X, there exists an isotopy $F: X \times [0,1] \to X$ with $F_0 = id_X$, $F_1(x) = y$. X is *isotopically representable* if for every x in X and u open in X such that $x \in u$, there is an open set v in X with the following properties: (1) $x \in v \subseteq u$; (2) if $y \in v$, there is an isotopy $F: X \times I \to X$ such that (a) $F_0 = id_X$, (b) $F_1(x) = y$, and (c) for every t in [0,1] and $z \notin v$, $F_t(z) = z$. X is *representable* if for every x in X and u open in X with $x \in u$, there is an open set v in X with the following property: if $y \in v$, there is an h in H(X) such that $h(x) = y$ and $h(z) = z$ for $z \notin v$.

A separable space X is said to be *countable dense homogeneous* if whenever A and B are countable dense subsets of X, there is some h in H(X) such that $h(A) = B$. If X is a continuum, then representable $\Rightarrow$ countable dense homogeneous $\Rightarrow$ n-homogeneous $\Rightarrow$ (n-1)-homogeneous ($n \in N$, $n \geq 2$), [BN, BT and U1].

The following will be needed [F]: Let h be a homeomorphism on a compact metric space X into a compact metric space Y, and let n be a positive integer. We define $\eta(h,n) = 2^{-n}\inf\{d'(h(x),h(y)) \mid x,y \in X \text{ and } d(x,y) \geq \frac{1}{n}\}$. (d denotes a metric on X compatible with its topology, and d' denotes a metric on Y compatible with its topology.)

Fort's Lemma. If $h_1, h_2, \cdots$ *is a sequence of homeomoprhisms of* X *onto itself such that* $\rho(h_n, h_{n+1}) < \eta(h_n, n)$ *for each* n*, then the sequence converges uniformly to a homeomorphism* h *of* X *to itself. (*ρ *denotes the sup metric in* H(X) *with respect to* d*.)*

A homeomorphism h in H(X) is *primitively stable* if h is the identity on some nonempty open set in X. A homeomorphism is *stable* if it is a composition of primitively stable homeomorphisms.

Suppose that R denotes the real numbers and $\{h_\alpha \mid \alpha \in R\}$ is a subcollection of H(X) with the following properties:

(1) For all α, $\beta \in R$, $h_{\alpha+\beta} = h_\alpha h_\beta$. (2) If $\Gamma: R \times X \to X$ is defined by $\Gamma(\alpha,x) = h_\alpha(x)$, then Γ is continuous. Then the subcollection $\hat{H} = \{h_\alpha | \alpha \in R\}$ will be called a flow in X. If $x \in X$, $\{h_\alpha(x) | h_\alpha \in \hat{H}\}$ is the orbit of x under $\hat{H}$, or $\hat{H}(x)$. The set of all points x of X with the property that $h_\alpha(x) = x$ for $\alpha \in R$ will be called the *invariant set* of $\hat{H}$.

The following result is by A. Beck [B]. It was given in the form below by James Keesling [K]:

Theorem (Beck). *Let* X *be a metric space and* F: R × X *be a flow on* X *with invariant set* S. *Then for any closed set* S' *containing* S *one can construct a new flow* F': $R \times X \to X$ *whose invariant set is* S'. *Moreover, for any* $x \in X - S'$ *with orbit* O(x) *under* F, *the orbit of* x *under* F' *is just the set of points which can be joined to* x *by an arc in* O(x) - S'. *Note that one may take the invariant set* S *to be the empty set.*

If G is a topological group and τ is a continuous homomorphism of the reals into G, then $\tau(R)$ is a *one parameter subgroup* of G. The following is well known. This form of it is due to J. Keesling [K]:

Lemma. If G *is a nontrivial connected locally compact topological group, then* G *has a nontrivial one parameter subgroup.*

3. Some Basic Results

Unless otherwise stated, X will denote a homogeneous continuum, d will denote a metric on X compatible with its topology, and ρ will denote the associated sup metric on H(X). If $\varepsilon > 0$, $k \in H(X)$, $N_\varepsilon(k) = \{h \in H(X) | \rho(h,k) < \varepsilon\}$.

Theorem 1. *H(X) is not locally compact.*

Proof. Assume H(X) is locally compact and u is an open subset of H(X) such that $1 \in u$; $\bar{u}$ is compact in H(X); and $u(x) \neq X$ for some $x \in X$. From a theorem of James Keesling [K], it follows that H(X) is zero-dimensional. Then there is some point q belonging to the boundary of u(x). There is a sequence $q_1, q_2, \cdots$ of u(x) such that $q_1, q_2, \cdots$ converges to q. For each i there is some h_i in u such that $h_i(x) = q_i$. Since $h_1, h_2, \cdots$ has some limit point h in u, some subsequence $h_{p_1}, h_{p_2}, \cdots$ of $h_1, h_2, \cdots$ converges to h. Then $h_{p_1}(x), h_{p_2}(x), \cdots$ converges to h(x) and since $q_{p_1}, q_{p_2}, \cdots$ converges to q, h(x) = 1.

This is a contradiction.

Theorem 2. *If x is a point in* X, *then* $H_x = H(X)_x$ *is an uncountable, closed subgroup of* H(X). (Note that it then follows that H_x is complete and dense in itself.)

Proof. It is well known that H_x is a closed subgroup of H(X). We need to prove that it is uncountable.

Suppose first that there is some point x in X such that H_x is discrete. Now $1 \in H_x$, and 1 is not a limit point of H_x, so there is $\varepsilon > 0$ such that $N_\varepsilon(1) \cap (H_x - \{1\}) = \emptyset$.

Suppose that $x_1, x_2, \cdots$ is a sequence of points of X which converges to x, and $\varepsilon_1, \varepsilon_2, \cdots$ is a decreasing sequence of positive numbers which converges to 0 such that for each i, $N_{\varepsilon_i}(1)$ contains a homeomorphism in $H_{x_i} - \{1\}$.

Now $N_{\varepsilon/16}(1)(x)$ is open in X, and there is some M in N such that if $m > M$, $N_{\varepsilon_m}(1)(x_m) \subseteq N_{\varepsilon/16}(1)(x)$ and $\varepsilon_M < \varepsilon/16$. For each $m > M$, there is some h_m in $N_{\varepsilon_m}(1)$ such that $h_m(x_m) = x_m$, but $h_m \neq 1$; and there is some k_m in $N_{\varepsilon/16}(1)$ such that $k_m(x) = x_m$. Then $k_m^{-1}h_m k_m \in N_{\varepsilon/4}(1)$. This is a contradiction, so there are no such sequences.

There is some $\alpha > 0$ such that if $z \in N_\alpha(1)(x)$, $\overline{N_\alpha(1)}$ contains no homeomorphism in $H_z - \{1\}$. Then $T_x \restriction N_{\alpha/2}(1)$ is a homeomorphism from $N_{\alpha/2}(1)$ onto $N_{\alpha/2}(1)(x)$, since it is both open and one-to-one.

Using translations of H(X), one easily sees that H(X) is locally homeomorphic to X. But then H(X) must be locally compact, a contradiction, so H_x is *not* discrete, and it must be infinite. But it must contain a limit point of itself, too, and so each point of H_x is a limit point of H_x, and, since it is closed, it must be uncountable.

The preceding theorem improves somewhat the result of G. Ungar [U2] and William Barit and Peter Renaud [BR] that there are no uniquely homogeneous continua. (A nondegenerate continuum is *uniquely homogeneous* if for each x and y in X, there is exactly one h in H(X) such that h(x) = y.) To see this, note that if x and y are in X, then there is some h in H(X) such that h(x) = y, and $hH_x = \{k \in H(X) | k(x) = y\}$. Thus this set is uncountable and dense in itself.

Remark. Suppose that H is a complete subgroup of H(X) with the property that H is transitive on X. Although

(H,X) is a polish topological transformation group with H transitive on X, it may well be the case that, with respect to H, X is uniquely homogeneous, i.e., it may be the case that if x, y are in X, there is exactly one h in H such that h(x) = y. However this would be the case if and only if X were itself a compact metric topological group (not necessarily a continuum). In fact, in that case X is homeomorphic to H.

Theorem 3. *Suppose* X *is an infinite compact metric space, and* D *is a complete infinite subgroup of* H(X). *If* $x \in X$ *such that* D(x) *is closed in* X, *and* $x' \in D(x)$, *then if* $D_{x'} \subseteq D_x$, $D_{x'} = D_x$.

Proof. Suppose that there is x' in D(x) such that $D_{x'} \subsetneq D_x$. Let $\mathcal{A} = \{D_z | D_z \subsetneq D_x \text{ and } z \in D(x)\} \cup \{D_x\}$. Now $\{D_{x'}, D_x\}$ is a monotonic subcollection of $\mathcal{A}$, so there is a maximal monotonic subcollection $\mathcal{B}$ of $\mathcal{A}$ that contains $\{D_{x'}, D_x\}$.

Since $\mathcal{B}$ is a collection of closed sets, $F = \{D - D_z | D_z \in \mathcal{B}\}$ is a collection of open sets, and some countable subcollection E of $\mathcal{B}$, which will be denoted by $\{D_{z_1}, D_{z_2}, \cdots\}$, has the property that $\{D - D_{z_i} | i \in N\}^* = F^*$. The sequence $D_{z_1}, D_{z_2}, \cdots$ has a subsequence $D_{y_1}, D_{y_2}, \cdots$ that is maximal with respect to the property that $D_{y_1} \supsetneq D_{y_2} \supsetneq \cdots$. Now if this subsequence is finite, a contradiction is reached immediately: There is some n in N such that $D_{y_1}, D_{y_2}, \cdots = D_{y_1}, D_{y_2}, \cdots, D_{y_n}$, and it follows that $\cap \mathcal{B} = D_{y_n}$. There is some ℓ in D such that $\ell(y_n) = x$. Then $\ell^{-1}D_x\ell = D_{y_n}$ and $\ell^{-1}D_{x'}\ell = D_s$ where s is the point of D(x) such that $\ell(s) = x'$. But then $D_{x'} \subsetneq D_x$ implies that $\ell^{-1}D_{x'}\ell \subsetneq \ell^{-1}D_x\ell$ and $D_s \subsetneq D_{y_n}$, which is a contradiction to the maximality of β.

Thus, assume that $D_{y_1}, D_{y_2}, \cdots$ is an infinite sequence. The same problem arises: Since X is compact, $y_1, y_2, \cdots$ must have a convergent subsequence $y_{p_1}, y_{p_2}, \cdots$. Call the limit of the subsequence $\hat{y}$. Note $\hat{y} \in D(x)$, and that Ancel's version of Effros's theorem can be applied to (D,D(x)).

Now if for some J in N, $D_{\hat{y}} \subsetneq D_{y_{p_J}}$, there is some h in $D_{\hat{y}} - D_{y_{p_J}}$. There is a sequence $h_1, h_2, \cdots$ of homeomorphisms of D which converges to h such that for each j in N, $h_j(\hat{y}) = y_{p_j}$. There is another sequence $k_1, k_2, \cdots$ of homeomorphisms in D which converges to 1 such that for each j in N, $k_j(y_{p_j}) = \hat{y}$. Then for each j, $h_jk_j(y_{p_j}) = y_{p_j}$, and $h_1k_1, h_2k_2, \cdots$ converges to h.

But there is an open set u in D such that $h \in u$ and $u \cap D_{y_{p_J}} = \emptyset$, which means that if $j' > J$, $h_{j'}k_{j'} \in D_{y_{pj}} \subseteq D_{y_{pJ}}$, so $h_jk_h \notin u$. This can't be. Then $D_{\hat{y}} \subseteq D_{y_{p_J}}$ for $j \in N$, and $\cap\beta = D_{\hat{y}}$, and, again, we have a contradiction. Then $D_x = D_{x'}$.

Theorem 4. *Suppose* X *is a homogeneous compact metric space. Suppose that* D *is a complete subgroup of* H(X) *that is transitive on* X, *and that* $x \in X$. *Let* $C_x = \{z \in X | D_x = D_z\}$. *Then* $E = \{hC_x | h \in D\}$ *is a partition of* X *into closed homeomorphic sets and, further, if* $y \in X$ *and* $h \in D$ *such that* $h(x) = y$, $hC_x = C_y$. *Also,* C_x, *considered as a subspace of* X, *is homogeneous (with respect to* D *as well as* H(X)*).* *Let* $G' = \{h \in D | h(C_x) = C_x\}$ *and* $G = \{h \upharpoonright C_x | h \in G'\}$. *Then* G' *is a closed subgroup of* D, G *is a compact subgroup of* $H(C_x)$, G *is homeomorphic to* C_x, *and* C_x *is uniquely homogeneous with respect to* G.

Proof. For convenience, let $C_x = C$. C is closed in X.

(1) Suppose h is in D such that $h(C) \cap C \neq \emptyset$. Then there is some c in C such that h(c) is in C. Let $h(c) = c'$. Now $hD_c h^{-1} = D_{c'} = hD_{c'}h^{-1} = D_c$. Suppose $z \in C$. Let $h(z) = z'$. Since $D_z = D_c$, $D_c = hD_c h^{-1} = hD_z h^{-1} = D_{z'}$. Thus $h(C) \subseteq C$. Likewise $h^{-1}(C) \subseteq C$, and so $h(C) = C$.

(2) If h and k are in D such that $h(C) \cap k(C) \neq \emptyset$, then $h(G) = k(C)$: Since $h(C) \cap k(C) \neq \emptyset$, $k^{-1}h(C) \cap C \neq \emptyset$, and $k^{-1}h(C) = C$, or $h(C) = k(C)$. Thus $E = \{h(C) | h \in D\}$ is a partition of X into homeomorphic closed sets.

(3) Suppose $y \in X$ and $h \in D$ such that $h(x) = y$. Then $hD_x h^{-1} = D_y$. If $z \in C_x$, $D_y = hD_z h^{-1} = D_{h(z)}$, and so $h(z) \in C_y$. Then $hC_x \subseteq C_y$. Similarly, $h^{-1}Cy \subseteq C_x$, and so $hC_x = C_y$.

(4) It is easy to see that G' is a closed subgroup of D, and it is also easy to see that G is a subgroup of $H(C_x)$.

Suppose c and c' are in $C_x = C$. By the definition of C, $e = 1 \restriction C_x$ is the only element of G which maps c to c (or any point c to itself). Then there is exactly one homeomorphism g in G such that $g(c) = c'$.

(5) Choose $c \in C$. Define $R_c \colon H(C) \to C$ by $R_c(f) = f(c)$ for $f \in H(C)$. Then $R_c \restriction G$ is a homeomorphism from G onto C:

From Effros's theorem we know that R_c is an open map from H(C) onto C. It is clear that $R_c | G$ is a one-to-one map from G onto C. But is it an open map?

Suppose o is an open subset of G. If $R_c(o)$ is not open, then there is some d in $R_c(o)$ which is not in its interior, i.e., there is a sequence $d_1, d_2, \cdots$ of X which converges to d such that for each i, $d_i \notin R_c(o)$. (Note: o and $R_c(o)$ must be uncountable.) Since $d \in R_c(o)$, there is some f in o such that $f(c) = d$. Now $f = f' \restriction C$ for some f' in D. For each i there is unique f_i in G such that $f_i(c) = d_i$, and there is some f_i' in D such that $f_i' \restriction C = f_i$. Now $f_1'(c), f_2'(c), \cdots$ converges to $f'(c) = d$, so there is a sequence $h_1, h_2, \cdots$ of D which converges to f' such that for each i, $h_i(c) = f_i'(c) = d$. Then $f_i' \restriction C = h_i \restriction C = f_i$ (since f_i unique), and $f_1, f_2, \cdots$ converges to f. But then eventually $f_i \in o$ and $f_i(c) = d_i \in R_c(o)$.

Thus, $R_c(o)$ is open, and $R_c \restriction G$ is an open map from G onto C. Since $R_c \restriction G$ is both open and one-to-one, $R_c \restriction G$ is a homeomorphism from G onto C.

Then the proof is finished, since it now follows that G and C are homeomorphic, and G is compact.

An application of Theorem 4 might be something like the following: Suppose X and Y are compact homogeneous metric spaces. Suppose further that Y is a topological group. Then $X \times Y$ is a compact homogeneous metric space, and $H(X \times Y)$ admits a complete subgroup D which is (A) transitive on $X \times Y$, and (B) algebraically and topologically equivalent to $H(X) \times Y$. If $(x,y) \in X \times Y$, $C_{(x,y)} = \{x\} \times Y$ and thus $E = \{\{x\} \times Y | x \in X\}$.

Theorem 4 also gives the following corollary.

Corollary 5. *Suppose* X *is a homogeneous continuum, and* $x \in X$. *Then the* C_x *of Theorem* 4 *is a nowhere dense compact subset of* X, *where* $D = H(X)$.

Proof. C_x cannot have interior unless $C_x = X$. But $C_x = X$ implies X is uniquely homogeneous.

Corollary 5 is really not very satisfying, because it seems to this author that one should be able to do much better. It seems that, in the case of $(H(X),X)$ with X a homogeneous continuum, C_x should be degenerate. But we have been unable to prove that that is so. Also, one should note here that if it does turn out that for every homogeneous continuum X, C_x is degenerate, then Corollary 5 and Theorem 9 (which appears later) become trivial. One might consider the following question:

Question. Suppose that X is a nondegenerate homogeneous continuum. If x and y are points of X, is there an h in H(X) such that $h(x) = x$, but $h(y) \neq y$?

Remark. If X is a homogeneous continuum, H(X) does *not* admit "small" open subgroups. This is a consequence of the following even more important fact, which is very easy to prove: Suppose o is a symmetric open subset of H(X) which contains 1. Then $D_o = \{f \in H(X) \mid$ for some n in N, $f = f_n \cdot f_{n-1} \cdots f_1$ where for each $i \leq n$, $f_i \in o\}$ is the closed-open subgroup of H(X) generated by o, and $D_o(x) = X$ for $x \in X$. (This also follows from results in [M].) Thus if u is an open subset of H(X) which contains D_o, there are limits on just how small u can be, since u(x) = X.

Theorem 6. *If* x *belongs to a homogeneous continuum* X*, there is a dense* G_δ *subset* E_x *of* X *such that if* $y \in E_x$*,* $H(X)_x(y)$ *is uncountable and dense in itself.*

Proof. Again, for convenience, let H = H(X) and $H_x = H(X)_x$. Suppose that $o_1, o_2, \cdots$ is a basis of symmetric open sets for 1, with $o_i \supseteq o_{i+1}$ for $i \in N$. Then $D_{o_1}, D_{o_2}, \cdots$ is a nested sequence of closed-open subgroups of H(X) with the property that D_{o_i} is transitive on X for each i. For each $i \in N$, apply the results of Theorem 4 to (D_{o_i}, X). Let C_i denote the C_x associated with (D_{o_i}, X). Then C_i is either nowhere dense, or C_i has interior and is X. But now $C_i = X$ means that C_i is uniquely homeomorphic with respect to D_{o_i}. This cannot be since $H_x \cap D_{o_i}$ is uncountable. Then C_i is nowhere dense and $E_x = X - \cup_{i=1}^{\infty} C_i$ is a dense G_δ subset of X.

Suppose $y \in E_x$. For each i there is $h_i \in o_i$ such that $h_i(y) \neq y$ and $h_i(x) = x$. (Otherwise $k \in D_{o_i}$ and $k(x) = x$

implies $k(y) = y$.) Then $H_x(y)$ is at least infinite. Since H_x is transitive on $H_x(y)$ and y is a limit point of $H_x(y)$, every point of $H_x(y)$ is a limit point of $H_x(y)$.

Suppose $H_x(y)$ is countably infinite. List the elements of $H_x(y)$: $y_o = y, y_1, y_2, \cdots$. For each i, let $A_i = \{h \in H_x | h(y) = y_i\}$. Then $\cup_{i=0}^{\infty} A_i = H_x$, which is an uncountable complete metric space. Then H_x is second category in itself and some A_i is second category in H_x. (See [KK] for a discussion of this.)

Now A_i is closed in H_x and $hA_o = A_i$ where h is some element of H_x such that $h(y) = y_i$. Then A_o has interior in H_x, and, likewise, so does each A_j. For each i, let $B_i = A_i^o$. But A_o is a group, so it is both closed and open in H_x; and the A_i's divide H_x up into a countable number of mutually exclusive open sets. In fact, it must be the case that $B_i = A_i$ for each i, and, specifically, $A_o \supseteq \mathcal{O}_n \cap H_x$ for some n. But this is a contradiction. $H_x(y)$ is uncountable.

Remark. If x and y are points of a homogeneous continuum, then H_x (defined as in Theorem 6) need not act micro-transitively on $H_x(y)$. For example, the dyadic solenoid Σ is a compact, connected abelian group containing a continuous homomorph D of the reals as a dense subgroup ([HR], p. 114). Then Beck's theorem (Section II) gives that if x and y are 2 points of D, $H_x(y)$ must be dense in Σ, and since it is a dense F_σ set in Σ it does *not* have a complete metric, and thus Ancel's version of Effros's theorem gives that H_x does not act micro-transitively on $H_x(y)$.

However, $H_x(y)$ does at least have to be a Borel subset of X, for X compact metric. (See [KK, E].)

If X is a homogeneous continuum, is there, in general, a class of subgroups of H(X) such that if G is one of those subgroups, and $x \in X$, then G acts micro-transitively on G(x)? One possible candidate might be the set of all closed, normal subgroups of H(X). This author doesn't know whether a closed, normal subgroup of X would have such a property or not, but does have the following:

Theorem 7. *Suppose* X *is a homogeneous, compact metric space,* H *is a complete subgroup of* H(X) *which is transitive on* X, *and* C *is a normal subgroup of* H. *Then* $\mathcal{C} = \{C(x) | x \in X\}$ *is a partition of* X *into homeomorphic sets. Further,* $\mathcal{D} = \{\overline{C(x)} | x \in X\}$ *continuously partitions* X *into closed, homeomorphic sets.*

Proof. Suppose first that C is a normal subgroup of H. Now if $x \in X$, $h \in H$, $hC(x) = Ch(x)$ since C is normal. Suppose $C(x) \cap C(y) \neq \emptyset$ for x, y in X. There is some ℓ in H such that $\ell(x) = y$, and thus $C(y) = C(\ell(x)) = \ell C(x)$. Then $C(x) \cap C(\ell(x)) \neq \emptyset$. There are k and k' in C such that $k(x) = k'\ell(x)$, and $k'^{-1}k(x) = \ell(x)$. Then $y \in C(x)$, and $C(y) \subseteq C(x)$. Likewise, $C(x) \subseteq C(y)$, and so $C(x) = C(y)$. We are done with the first part.

It is also the case that $\mathcal{D} = \{\overline{C(x)} | x \in X\}$ is a partition of X into closed homeomorphic sets:

If $z \in \overline{C(x)}$, $C(z) \subseteq \overline{C(x)}$. Then $\overline{C(x)}$ is a union of sets from $\mathcal{C}$. Suppose $z \in X$ such that $\overline{C(x)} \cap \overline{C(z)} \neq \emptyset$. If $y \in \overline{C(x)} \cap \overline{C(z)}$, $\overline{C(y)} \subseteq \overline{C(x)} \cap \overline{C(z)}$.

Suppose $\overline{C(y)} \subsetneq \overline{C(x)}$. Let $\mathcal{D} = \{\overline{C(r)} \mid r \in X\}$. Since $\{\overline{C(y)}, \overline{C(x)}\}$ is a monotone subcollection, there is a maximal monotone subcollection $\mathcal{B}$ of $\mathcal{D}$ which contains $\{\overline{C(y)}, \overline{C(x)}\}$. But $\bigcap_{\mathcal{B}} \overline{C(s)} \neq \emptyset$, so there is some point $t \in \bigcap_{\mathcal{B}} \overline{C(s)}$ and $\overline{C(t)} \subseteq \bigcap_{\mathcal{B}} \overline{C(s)}$. There is some ℓ in H such that $\ell(x) = t$ and so $\overline{C(t)} = \overline{C(\ell(x))} = \ell\overline{C(x)}$. Then $\overline{C(\ell(y))} = \ell\overline{C(y)} \subsetneq \ell\overline{C(x)} = \overline{C(\ell(x))} = \overline{C(t)}$. However, this contradicts the maximality of $\mathcal{B}$.

Well, then $\overline{C(y)} = \overline{C(x)}$. Likewise $\overline{C(y)} = \overline{C(z)}$, and $\overline{C(z)} = \overline{C(x)}$. We have a partition $\mathcal{D}$ of X into homeomorphic closed sets.

From Theorem 4 of [R] it follows that $\mathcal{D}$ continuously partitions, or decomposes X.

Theorem 8. *Suppose* X *is a homogeneous compact metric space,* H *is a complete subgroup of* H(X) *which acts transitively on* X, *and* $\mathcal{C}$ *is a collection of subsets of* X *such that* (1) *if* $C, D \in \mathcal{C}$ *and* $h \in H$, *then* $hC \cap D \neq \emptyset$ *implies* $hC = D$, (2) $\mathcal{C}^* = X$. (In other words, $\mathcal{C}$ is a partition of X into homeomorphic sets which H respects.) *If* $x \in X$, *let* C(x) *denote the element of* $\mathcal{C}$ *which contains* x, *and let* $R_x = \{h \in H \mid h(x) \in C(x)\}$. *Then if* $x, y \in X$, R_x *is a subgroup of* H *and* R_x *and* R_y *are conjugate,* i.e., *there is* h *in* H *such that* $R_x = h^{-1}R_y h$. *Further, if the elements of* $\mathcal{C}$ *are closed subsets of* X, *then for* $x \in X$, R_x *is closed in* H.

Proof. Of course, $\mathcal{C}$ defines an equivalence relation on X. Fix $x \in X$. Now $1 \in R_x$. If $h \in H$ and $z \in X$, then since $h(z) \in hC(z) \cap C(h(z))$, $hC(z) = C(h(z))$. If $h \in R_x$, $C(x) = hC(x) = C(h(x))$, and $h^{-1}C(x) = h^{-1}(hC(x)) = C(x)$, so $h^{-1}(x) \in C(x)$ and $h^{-1} \in R_x$. If $g, h \in R_x$, $h^{-1} \in R_x$ and $gh^{-1}(x) \in gh^{-1}C(x) = gC(x) = C(x)$. Thus, R_x is a subgroup.

Suppose x,y are in X. There is h in H such that $h(x) = y$. Then $h^{-1}R_y h$ is a subgroup of H, and further $h^{-1}R_y h = R_x$: If $\ell \in R_y$, $\ell(y) \in C(y)$. Then $h^{-1}\ell h(x) \in h^{-1}\ell hC(x) = h^{-1}\ell Ch(x) = h^{-1}\ell C(y) = h^{-1}C(\ell(y)) = h^{-1}C(y) = C(h^{-1}(y)) = C(x)$ and $h^{-1}\ell h \in R_x$. Likewise, $hR_x h^{-1} \subseteq R_y$. Then $h^{-1}R_y h \subseteq R_x$ means $R_y \subseteq hR_x h^{-1}$, and $R_y = hR_x h^{-1}$.

Suppose $x \in X$ and $C(x)$ is closed in X. If $h_1, h_2, \cdots$ is a sequence of homeomorphisms in R_x which converges to h, then $h_1(x), h_2(x) \cdots$ converges to $h(x)$ in X and since $h_i(x) \in C(x)$ for each i, $h(x) \in C(x)$. Thus $h \in R_x$, and R_x is closed.

Note. It is not possible to improve the preceding to get that the R_x's are normal. For example, take H(X) to be H, and $\mathcal{C}$ to be the trivial partition of X, i.e., $\mathcal{C} = \{\{x\} | x \in X\}$. Then $R_x = H(X)_x$ which is not normal in H(X).

For more information on partitions of homogeneous continua, one might read [R], noting in particular Theorem 4 from that paper.

Suppose that X is a homogeneous continuum. Below we list some of the more obvious normal subgroups of H(X) that one might want to consider:

(1) The Center $C = \{h \in H(X) \mid \text{if } f \in H(X), hf = fh\}$. Is C always degenerate? It is quite easy to check that it is if X is 2-homogeneous, has the fixed point property, or has property *. [To say that X has property * means that if M is a subcontinuum of X, p, q are points of M, and $\varepsilon > 0$, there exists a homeomorphism $h \in H(X)$ such that $h(p) = q$ and $h(z) = z$ for each z outside the ε-neighborhood of M. Wayne Lewis [L1] showed that the pseudo-arc has property *. Clearly any representable continuum has this property.] This author doesn't know the answer to the general question, but does have the theorem below. C is, at most, a compact, abelian, totally disconnected, nowhere dense subgroup of H(X). Thus, we have a certain basis for the claim that H(X) must very much fail to be abelian.

(2) The commutator $Q = \{aba^{-1}b^{-1} \mid a,b \in H(X)\}$. If $x \in X$, is it always the case that $Q(x) = X$? If X is 2-homogeneous or has the fixed point property then it is easy to check that $Q(x) = X$. This particular subgroup proved to be rather useful in [P1].

(3) $G_1 = \{g \in H(X) \mid g \text{ is stable}\}$.

(4) $G_2 = \{g \in H(X) \mid g$ is a finite composition or homeomorphisms, each of which leaves some point of X fixed$\}$.

(5) G_3 = the component of the identity.

(6) G_4 = the arc-component of the identity.

(7) $G_5 = \{g \in H(X) \mid$ there is a continuum C from 1 to g in H(X)$\}$.

(8) $G = \cap_i D_{\mathcal{O}_i}$ where (1) $\mathcal{O}_1, \mathcal{O}_2, \cdots$ is a basis of symmetric open sets for 1, with $\mathcal{O}_i \supseteq \overline{\mathcal{O}_{i+1}}$ for $i \in N$, (2) $D_{\mathcal{O}_i}$

is the closed-open subgroup generated by o_i (see Theorem 6). If $x \in X$, is $G(x) = X$? Must G at least be nondegenerate?

Theorem 9. *If* X *is a homogeneous continuum and* C *denotes the center of* H(X), C *is a compact, totally disconnected, nowhere dense subgroup of* H(X).

Proof. If $f \in C$ and $x \in X$, then $f^{-1}H_x f = H_x = H(X)_x$. But also $f^{-1}H_x f = H_{f^{-1}(x)}$. Define $F_x = \{z \in X | H_x = H_z\}$ and $G_x = \{f \in H(X) | f(F_x) = F_x\}$. From Corollary 5, F_x is a nowhere dense compact subset of X. Now $C \subseteq G_x$, for $f \in C$ implies $H_x = H_{f(x)}$ and $f(x) \in F_x$. Then $f(F_x) \cap F_x \neq \emptyset$ and $f(F_x) = F_x$. C is by definition abelian and since G_x is nowhere dense in H(X), C is nowhere dense. Also, C is closed in H(X).

Note that $C \subseteq G_y$ for each y in X. Define $D_x = \{f \upharpoonright F_x | f \in G_x\}$. In theorem 4, we proved that D_x is homeomorphic to F_x. Now C is homeomorphic to a subgroup of D_x: Define $\phi: C \to D_x$ as follows: $\phi(f) = f|F_x$ for $f \in C$. ϕ is clearly continuous. ϕ is also one-to-one: Suppose $f,g \in C$. If $\phi(f) = \phi(g)$, then $z \in F_x$ implies $f(z) = g(z)$, and specifically, $f(x) = g(x)$. Suppose y is a point in X. There is some α in H(X) such that $\alpha(x) = y$. Then $\alpha f(x) = \alpha g(x) = f\alpha(x) = f(y) = g\alpha(x) = g(y)$, and $f = g$. ϕ is also reversibly continuous: Suppose $f_1 \upharpoonright F_x, f_2 \upharpoonright F_x, \cdots$ is a sequence in $\phi(C)$ which converges to $f \upharpoonright F_x$ in $\phi(C)$. Then $f_1(x), f_2(x), \cdots$ converges to $f(x)$. Suppose $y \in X$, and $y_1, y_2, \cdots$ converges to y. There is a sequence $\alpha_1, \alpha_2, \cdots$ of homeomorphisms of H(X) which converges to a homeomorphism α of H(X) such that $\alpha_i(x) = y_i$ for each i and $\alpha(x) = y$.

Then $\alpha_1 f_1(x), \alpha_2 f_2(x), \cdots$ converges to $\alpha f(x)$ and since $\alpha_i f_i(x) = f_i \alpha_i(x)$ for each i and $\alpha f(x) = f\alpha(x)$, we have that $f_1(y_1), f_2(y_2), \cdots$ converges to $f(y)$. Then $f_1, f_2, \cdots$ converges to f. So C and $\phi(C)$ are homeomorphic. Also $\phi(C)$ is a group. Since C is complete, $\phi(C)$ is complete (considered as space), which means that $\overline{\phi(C)} = \phi(C)$. Then $\phi(C)$ is compact, and so is C. Also $C(x) = \overline{C(x)}$ for $x \in X$.

Let $\mathcal{E} = \{C(x) | x \in X\}$. $\mathcal{E}$ is a partition of X into closed homeomorphic sets with the property that if $k, h \in H(X)$, $y \in X$, $hC(y) \cap kC(y) \neq \emptyset$ means $hC(y) = kC(y)$.

We must show that C is totally disconnected. Suppose not; suppose C contains a nondegenerate connected set. Let C_o denote the component of 1 in C. From one of the lemmas in Section 2, we know then that C has a non-trivial one parameter subgroup $\Gamma: R \to C_o$ (where R denotes the reals, Γ is a continuous homomorphism from R in C_o). Then $\gamma: R \times X \to X$ defined by $\gamma(t,x) = \Gamma(t)(x)$ is continuous and is a non-trivial flow in X. Now the invariant set S of γ is ϕ: There is some t such that $\Gamma(t) \neq 1$ and since if $x \in X$, $\Gamma(t)$ moves everything in $C(x)$, $\Gamma(t)$ moves everything in X.

Fix $x \in X$, pick $f \in C_o - \{1\}$. Then suppose o is an open set in X such that (1) $x \in o$, (2) $f(x) \notin \overline{o}$. There is a new flow $\gamma': R \times X \to X$ with invariant set $S' = X - o$ and the property that if A is an arc in o such that $A \subseteq M$ where $M = \{\gamma_t(x) | t \in R\} - S'$, and $z \in A - \{x\}$, there is $t \in R$ such that $\gamma'_t(x) = z$. But then $f\gamma'_t(x) = f(z)$ and $\gamma'_t f(x) = f(x)$. This is a contradiction: $f\gamma'_t(x) = \gamma'_t f(x)$ but $x \neq z$

implies $f(x) \neq f(z)$.

C does not contain a nondegenerate connected set.

4. Results Involving Stronger Homogeneity Properties

Again, unless otherwise stated, X will denote a homogeneous continuum, d will denote a metric on X compatible with its topology, and ρ will denote the associated sup metric on H(X). If $\varepsilon > 0$, $k \in H(X)$, $N_\varepsilon(k) = \{h \in H(X) | \rho(h,k) < \varepsilon\}$.

Next we extend a result that appeared in [P1]. There it was proved that a product of homeotopically homogeneous continua is representable.

First we need a lemma.

Lemma 10. *Suppose* X *is a homeotopically homogeneous continuum. Let* $A = \{F: [0,1] \to H(X) | F(0) = 1$ *and* F *is continuous*$\}$. *Then* (1) A *is a complete, metric, separable, arcwise connected, locally arcwise connected group, and* (2) *if* $x \in X$ *and* $\varepsilon > 0$, *there is some* $\delta > 0$ *such that if* $y \in D_\delta(x) = \{z \in X | d(x,z) < \delta\}$, *then there is a path* $F \in A$ *such that* $F_0 = 1$, $F_1(x) = y$, *and* $d(F_t(z),z) < \varepsilon$ *if for each* $(z,t) \in X \times [0,1]$.

Proof. Let $A' = \{F: [0,1] \to H(X) | F$ is continuous$\}$. Put on A' the usual sup metric $\hat{\rho}$ (i.e., if F, G $\in$ A', $\hat{\rho}(F,G) = \text{lub}\{\rho(F_t,G_t) | t \in [0,1]\}$). Now A' with this metric is a complete separable metric space. (This is well known: see, for example [KK].)

Now $A \subseteq A'$, and A is closed in A'. Further A is a topological group under the composition operation. Thus A is a complete separable metric topological group.

A is arcwise connected: Suppose $F \in A$. Define $\phi: [0,1] \to A$ as follows: If $t, \hat{t} \in [0,1]$, $\phi(t)_{\hat{t}} = F_{t \cdot \hat{t}}$. Evidently ϕ is at least well-defined. Is ϕ continuous? Suppose $t_1, t_2, \cdots$ converges to t in $[0,1]$. Then if $\hat{t}_1, \hat{t}_1, \cdots$ converges to $\hat{t}$ in $[0,1]$, $t_1\hat{t}_1, t_2\hat{t}_2$, converges to $t\hat{t}$ in $[0,1]$, and $F_{t_1\hat{t}_2}, \cdots$ converges to $F_{t\hat{t}}$. Then $\phi(t_1)_{\hat{t}_1}, \phi(t_2)_{\hat{t}_2}, \cdots$ converges to $\phi(t)_{\hat{t}}$ and $\phi(t_1), \phi(t_2), \cdots$ converges to $\phi(t)$. Then there is a path from 1_A to F (where 1_A is the identity for A), and there is an arc from 1_A to F.

It then follows that A is locally arcwise connected by Theorem 3.7 of Ungar in [U2]. Then (A,X) is a polish topological transformation group when the action ϕ of A on X is defined by $\phi(F,x) = F_1(x)$ for $(F,x) \in A \times X$. Further, A acts transitively on X, and thus Effros's theorem can be applied.

We need to prove (2). Suppose $x \in X$ and $\varepsilon > 0$. Let $\hat{N}_\varepsilon(1_A) = \{F \in A \mid \hat{\rho}(F,1_A) < \varepsilon\}$, and $\hat{N}_\varepsilon(1_A)(x) = \{z \in X \mid z = F_1(x)$ for $F \in \hat{N}_\varepsilon(1_A)\}$. There is some connected open subset o of A such that $1_A \in o \subseteq \hat{N}_\varepsilon(1_A)$. Then $x \in o(x)$ (= $\{z \in X \mid$ there is $F \in o$ such that $F_1(x) = z\}$) which is open in X. Suppose $z \in o(x)$. There is some G in o such that $G_1(x) = z$. Then $G \in \hat{N}_\varepsilon(1_A)$. Then G is the desired path.

Theorem 11. *Suppose* X *and* Y *are homeotopically homogeneous continua. Then* $X \times Y$ *is isotopically representable.*

Proof. Let $A_X = \{F: [0,1] \to H(X) \mid F$ is continuous and $F_o = 1_X\}$ and $A = \{F: [0,1] \to H(Y) \mid F$ is continuous and $F_o = 1_Y\}$.

Suppose $(x,y) \in X \times Y$ and $u = u_X \times u_Y$ is open in $X \times Y$. In this proof, we will abuse notation somewhat and let d denote both a metric on X compatible with its topology, and a metric on Y compatible with its topology. There is $\varepsilon > 0$ such that $\overline{D_\varepsilon(x)} \subseteq u_X$ and $\overline{D_\varepsilon(y)} \subseteq u_Y$ (if $z \in X$, $\delta > 0$, $D_\delta(z) = \{t \in X | d(t,z) < \delta\}$). There is some positive number $\hat{\delta} < \varepsilon/8$ such that if $t \in D_{\hat{\delta}}(y)$, there is $G\colon [0,1] \to H(Y)$ in A_Y such that $G_0 = 1$, $G_1(y) = t$, $d(G_{\hat{t}}(w),w) < \varepsilon/4$ for $(w,\hat{t}) \in Y \times [0,1]$. Pick $y' \in D_{\hat{\delta}}(y) - y$. There is some open set $V(2)$ such that $y \in V(2) \subseteq \overline{V(2)} \subseteq (D_{\hat{\delta}}(y) - y) \cap u_Y$.

There is some $\hat{\varepsilon} > 0$ such that $D_{\hat{\varepsilon}}(y) \subseteq v(2)$. There is $G'\colon [0,1] \to H(Y)$ in A_Y such that $G'_0 = 1_Y$, $G'_1(y) = y'$, $d(G'_{\hat{t}}(w),w) < \varepsilon/4$ for $(w,\hat{t}) \in Y \times [0,1]$.

Now there is some positive number $\delta < \varepsilon/8$ such that if $z \in D_\delta(x)$, there is an $F\colon [0,1] \to H(X)$ in A_X such that $F_0 = 1_X$, $F_1(x) = z$ and $d(F_t(w),w) < \varepsilon/4$ for $(w,t) \in X \times [0,1]$. Pick $x' \in D_\delta(x)$. There is an $F'\colon [0,1] \to H(X)$ in A_X such that $F'_0 = 1_X$, $F'_1(x) = x'$ and $d(F'_t(w),w) < \varepsilon/4$ for $(w,t) \in X \times [0,1]$. Define $\Gamma\colon Y \to [0,1]$ as follows: $\Gamma(b) = \dfrac{\hat{\varepsilon}-\min\{d(y,b),\hat{\varepsilon}\}}{\hat{\varepsilon}}$ for $b \in Y$. Define $\phi\colon X \times Y \to X \times Y$ as follows: If $(a,b) \in X \times Y$, $\phi(a,b) = (F'_{\Gamma(b)}(a),b)$. Now if $b \notin v(2)$, $\Gamma(b) = 0$ and $\phi(a,b) = (a,b)$. Also, $\phi(x,y) = (F'_1(x),y) = (x',y)$. Note that ϕ is a homeomorphism.

Then define $\Omega\colon X \times Y \times [0,1] \to X \times Y$ as follows: if $(a,b,t) \in X \times Y \times [0,1]$, $\Omega(a,b,t) = (F'_{t\Gamma(b)}(a),b)$. Then Ω is a path of homeomorphisms in $H(X \times Y)$, and (1) $\Omega_0 = 1_{X\times Y}$, (2) $\Omega(x,y,1) = (F'_{\Gamma(y)}(x),y) = (x',y)$.

There is a positive number $\hat{\hat{\delta}} < \varepsilon/8$ such that $D_{\hat{\hat{\delta}}}(x') \subseteq D_\delta(x)$. Define $\beta\colon X \to [0,1]$ as follows:

$\beta(a) = \frac{\hat{\hat{\delta}}-\min\{d(a,x'),\hat{\hat{\delta}}\}}{\hat{\hat{\delta}}}$ for $a \in X$. Define $K: X \times Y \to X \times Y$ as follows: $K(a,b) = (a,G'_{\beta(a)}(b))$ for $(a,b) \in X \times Y$. K is a homeomorphism; if $a \notin D_{\hat{\delta}}(x')$, $\beta(a) = 0$ and $K(a,b) = (a,b)$ for any $b \in Y$. Also, $K(x',y) = (x',y')$.

Define $\kappa: X \times Y \times [0,1] \to X \times Y$ by $\kappa(a,b,t) = (a,G'_{t\beta(a)}(b))$. Then κ is a path of homeomorphisms such that (1) $\kappa_0 = 1_{X\times Y}$, (2) $\kappa(x',y,1) = (x',y')$.

Define $C: X \times Y \times [0,1] \to X \times Y$ as follows: if $(a,b,t) \in X \times Y \times [0,1]$, $C(a,b,t) = \kappa_t^{-1} \circ \Omega_t^{-1} \circ \kappa_t \circ \Omega_t(a,b)$. Then C is a path of homeomorphisms such that (1) $C_0 = 1_{X\times Y}$; (2) $C(x,y,1) = (x',y)$, and (3) if $(a,b) \in X \times Y$ such that $(a,b) \notin D_\varepsilon(x) \times D_\varepsilon(y)$, then $C(a,b,t) = (a,b)$ for any t in $[0,1]$.

Similarly, one can construct a path of homeomorphisms $D: X \times Y \times [0,1] \to X \times Y$ such that (1) $D_0 = 1_{X\times Y}$, (2) $D_1(c',y) = (x',y')$, and (3) $D_t(a,b) = (a,b)$ for $(a,b) \in X \times Y - u$, and $t \in [0,1]$.

Then $D \circ C: X \times Y \times [0,1] \to X \times Y$ is a path of homeomorphisms such that (1) $D_0 \circ C_0 = 1_{X\times Y}$, (2) $D_1 \circ C_1(x,y) = (x',y')$; and (3) $D_t \circ C_t(a,b) = (a,b)$ for $(a,b,t) \in (X \times Y - u) \times [0,1]$.

Note that it follows now that any finite or countably infinite product of homeotopically homogeneous continua is isotopically representable. One might wonder whether or not a homeotopically homogeneous continuum is isotopically representable.

In general, stable homeomorphisms seem to be (A) a big help when working with homogeneity properties; and (B) rather hard to come by, at least from the standpoint of proving that a member of a given class of homogeneous continua admits them, unless that is already obvious. However, when isotopies and products are involved, it is quite easy to get stable isotopies: One can quite easily prove, using much simplified techniques from the last proof, the following:

(1) If X and Y are continua and X admits a non-trivial isotopy, then $X \times Y$ admits stable homeomorphisms, and, in fact, stable isotopies.

(2) If X and Y are continua, X is homogeneous and admits a non-trivial isotopy, then $X \times Y$ is an isotopy Galois space.

In fact, when spaces admit isotopies, homogeneity properties seem to actually get stronger with the taking of products. The best example of this is the Hilbert cube: it is a product of nonhomogeneous spaces which turns out not only to be homogeneous but to have nearly every nice homogeneity property imaginable. With spaces which do *not* admit isotopies, the reverse seems to be true: Take a product with one of these as a factor, and no matter how nice in respects other than the isotopy one it is, the product seems to lose most of those nice properties: An important example here would be a product involving the Menger universal curve M. W. and K. Kuperberg and W. R. R. Transue proved in [KKT] that $M \times M$ is not 2-homogeneous, and $M \times S$ is not 2-homogeneous (S denotes the simple closed curve). Note that M is representable, and, thus, admits very nice stable homeomorphisms.

In [P1] it was shown that $M \times X$ is not 2-homogeneous, no matter what continuum X is. Note that $M \times S$ is, however, at least isotopy Galois. Wayne Lewis [L2] has recently shown that if X is a continuum, and $M \times X$ is Galois, then X is isotopy Galois. One might wonder whether if $X \times Y$ has one of the "nicer" homogeneity properties, it must be the case that X or Y admits isotopies. Also one might note that H(M) is totally disconnected [BB].

The following is true, however, and quite easy to prove, although a little messy.

Theorem 12. *Suppose that* $n \in N - \{1\}$ *or* $n = \omega$ *and* $m \in N - \{1\}$. *If* X *is a continuum such that* X^2 *is* m-*homogeneous* (countable dense homogeneous, representable) *then* X^n *is* m-*homogeneous* (countable dense homogeneous, representable).

Proof. Suppose first that $n \in N - \{1\}$. Now if X^2 is m-homogeneous ($m \geq 2$) then X^2 is strongly m-homogeneous, since X^2 is not a simple closed curve [U2].

Suppose that $m = 2$, $\{a_1, a_2\}$ and $\{b_1, b_2\}$ are two 2-point subsets of X^n ($n \in N - \{1,2\}$), and $\{c_1, c_2\}$ is a 2-point subset of X^n with the property that for each i, $c_1(i) \neq c_2(i)$. (If $x \in X^n$, denote x by $(x(1), x(2), \cdots, x(n))$.)

Now $a_1 \neq a_2$, so there is j_1 such that $a_1(j_1) \neq a_2(j_1)$. Let j_2 be an element of $\{1, \cdots n\}$ other than j_1. There is h_1 in $H(X^2)$ such that $h_1(a_1(j_1), a_1(j_2)) = (c_1(j_1), c_1(j_2))$ and $h_1(a_2(j_1), a_2(j_2)) = (c_2(j_1), c_2(j_2))$. For $x \in X^n$, define $\hat{h}_1\colon X^n \to X^n$ by $\hat{h}_1(x) = z$ where $(z(j_1), z(j_2)) = h_1(x(j_1), x(j_2))$ and for $i \notin \{j_1, j_2\}$, $z(i) = x(i)$. Then

$\hat{h}_1 \in H(X^n)$. There is a least $j_3 \notin \{j_1,j_2\}$ and there is some h_2 in $H(X^2)$ such that $h_2(c_1(j_1),a_1(j_3)) = (c_1(j_1),c_1(j_3))$ and $h_2(c_2(j_1),a_2(j_3)) = (c_2(j_1),c_2(j_3))$. For $x \in X^n$ define $\hat{h}_2\colon X^n \to X^n$ by $\hat{h}_2(x) = z$ where $(z(j_1),z(j_3)) = h_2(x(j_1),x(j_3))$ and $x(i) = z(i)$ for $i \notin \{j_1,j_3\}$. If $n = 3$, we are finished, for $\hat{h}_2 \circ \hat{h}_1(a_1,a_2) = (c_1,c_2)$; otherwise continue the preceding process n-1 times until $\{\hat{h}_1,\hat{h}_2,\cdots\hat{h}_{n-1}\}$ has been found such that $\hat{h}_{n-1}\hat{h}_{n-2}\cdots\hat{h}_2\hat{h}_1(a_1) = c_1$, and $\hat{h}_{n-1}\hat{h}_{n-2}\cdots\hat{h}_1(a_2) = c_2$. Let $h = \hat{h}_{n-1}\cdots\hat{h}_1$. Likewise, there is a homeomorphism k from X^n onto X^n such that $k(b_1,b_2) = (c_1,c_2)$. Then $k^{-1}h(a_1,a_2) = (b_1,b_2)$ and X^n is 2-homogeneous.

We use induction. Suppose that $m \in N - \{1,2\}$ such that X^2 is m-homogeneous, and it is already known that X^n is (m-1)-homogeneous. Then suppose that $\{a_1,\cdots a_m\}$ and $\{b_1,\cdots b_m\}$ are m-element subsets of X^n and $\{c_1,c_2,\cdots c_m\}$ is an m-element subset of X^n such that for each $i,\{c_1,(i),\cdots c_m(i)\}$ is an m-element subset of X. There is some h in X^n such that $h(a_1,a_2,\cdots a_{m-1}) = (c_1,c_2,\cdots c_{m-1})$. There are integers j_1,j_2 such that $\{\pi_{j_1j_2}(c_1),\pi_{j_1j_2}(c_2),\cdots,\pi_{j_1j_2}(c_{m-1}), \pi_{j_1j_2}(h(a_m))\}$ is an m-element subset of X^2. (If i,j are positive integers less than or equal to n, then $\pi_{ij}(x) = (x(i),x(j))$ for $x \in X^n$.)

Then k_1 can be chosen from $H(X^2)$ such that $k_1(\pi_{j_1j_2}(c_1),\pi_{j_1j_2}(c_2)),\cdots\pi_{j_1j_2}(c_{m-1}),\pi_{j_1j_2}(h(a_m))) = (\pi_{j_1j_2}(c_1),\cdots,\pi_{j_1j_2}(c_{m-1}),\pi_{j_1j_2}(c_m))$. For $x \in X^n$, define $\hat{k}_1 \in H(X^n)$ by $\hat{k}_1(x) = z$ where $(z(j_1),z(j_2)) = k_1(x(j_1),x(j_2))$ and for $i \notin \{j_1,j_2\}$, $z(i) = x(i)$. Thus, $\hat{k}_1(c_j) = c_j$ for

$j < m$. There is least $j_3 \notin \{j_1, j_2\}$. Now

$$\{\pi_{j_1 j_3}(c_1), \pi_{j_1 j_3}(c_2), \cdots \pi_{j_1 j_3}(c_{m-1}), \pi_{j_1 j_3}(\hat{k}_1 h(a_m))\}$$

is an m-element subset of X^2. (Note that $\hat{k}_1 h(a_m)(j_2) = c_m(j_2)$.) There is some k_2 in $H(X^2)$ such that $k_2(\pi_{j_1 j_3}(c_1), \cdots \pi_{j_1 j_3}(c_{m-1}), \pi_{j_1 j_3}(\hat{k}_1 h(a_m))) = (\pi_{j_1 j_3}(c_1), \cdots, \pi_{j_1 j_3}(c_m))$. Define $\hat{k}_2$ in $H(X^n)$ as follows: if $x \in X^n$, $\hat{k}_2(x) = z$ where $(z(j_1), z(j_3)) = k_2(x(j_1), x(j_3))$ and if $i \notin \{j_1, j_3\}$, $x(i) = z(i)$. If $n = 3$, we are done; otherwise continue this process $n - 1$ times obtaining $\{\hat{k}_1, \hat{k}_2 \cdots, \hat{k}_{n-1}\}$ such that $\hat{k}_{n-1} \circ \cdots \hat{k}_2 \hat{k}_1(a_1, a_2, \cdots a_m) = (c_1, c_2, \cdots c_m)$. Let $k = \hat{k}_{n-1} \cdots \hat{k}_1$. Likewise there is some ℓ in $H(X^n)$ such that $\ell(b_i) = c_i$ for each $i \leq m$. Then $\ell^{-1}k$ is the desired homeomorphism.

The first part is proved for $n \in N - \{1\}$. Suppose $n = \omega$ and X^2 is m-homogeneous, $m \geq 2$. Suppose d is a metric on X with $d(x,y) \leq 1$ for $(x,y) \in X^2$. Suppose e is the following metric on X^ω: $e(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} d(x(i), y(i))$. It is well known that e is a metric on X^ω compatible with its topology. Also let ρ denote the sup metric on $H(X^\omega)$ with respect to e. Suppose $m \geq 2$, and $\{a_1, a_2, \cdots a_m\}$ and $\{b_1, \cdots b_m\}$ are m-element subsets of X^ω. There are m-1 positive integers $j_1, j_2, \cdots j_{m-1}$ such that $\{s(a_1), s(a_2), \cdots, s(a_m)\}$ is an m-element subset of X^{m-1} (where $s = \pi_{j_1 j_2 \cdots j_{m-1}}$). Suppose that $\{c_1, \cdots c_m\}$ is an m-element subset of X^{m-1} such that for $i \leq m-1$, $\{c_1(i), \cdots c_m(i)\}$ is an m-element subset of X. Then there is some h in $H(X^{m-1})$ such that $h(s(a_i)) = c_i$ for each $i \leq m$. Define $h \in H(X^\omega)$

as before (or rather analogously).

We will use Fort's lemma. Suppose ℓ_1 is the first integer not in $\{j_1,\cdots j_{m-1}\}$. Then $\{\pi_{j_1\ell_1}(\hat{h}(a_1)),\cdots \pi_{j_1\ell_1}(\hat{h}(a_m))\}$ is an m-element subset of X^2. There is some $\{c_{11},c_{21},\cdots c_{m1}\}$, an m-element subset of $\pi_{j_1\ell_1}(X^\omega)$ $(\approx X^2)$ with the following properties:

(1) for $i = j_1$ or ℓ_1, $\{c_{11}(i),\cdots c_{m1}(i)\}$ is an m-element subset of X; and (2) $(c_{11},\cdots,c_{m1}) \in \pi_{j_1\ell_1}(N_{1/2}(1)(\hat{h}(a_1)\cdots \hat{h}(a_m)))$. There is $k_1 \in H(X^m)$ such that $\rho_X^m(k_1,1_{X^m}) < 1/2$ such that $k_1(\pi_{j_1\ell_1}(h(a_1)),\cdots\pi_{j_1\ell_1}(h(a_m))) = (c_{11},\cdots c_{m1})$. Define $\hat{k}_1$ as before--so that $\hat{k}_1 \in H(X^\omega)$. There is least $\ell_2 \not\in \{j_1,\cdots j_{m-1},\ell_1\}$. Then $\{\pi_{\ell_1\ell_2}(\hat{k}_1\hat{h}(a_1),\cdots\pi_{\ell_1\ell_2}(\hat{k}_1\hat{h}(a_m))\}$ is an m-element subset of $\pi_{\ell_1\ell_2}(X^\omega)$. There is some m-element subset $(c_{12},c_{22},\cdots c_{m2}\}$ of $\pi_{\ell_1\ell_2}(X^\omega)$ with the following properties: (1) for $i = \ell_1$ or ℓ_2, $\{c_{12}(i),\cdots c_{m2}(i)\}$ is an m-element subset of X, and (2) $(c_{12},\cdots,c_{m2}) \in \pi_{\ell_1\ell_2}(N_{\varepsilon_2}(1)(\hat{k}_1\hat{h}(a_1),\cdots\hat{k}_1\hat{h}(a_m)))$ where $\varepsilon_2 < \eta(\hat{k}_1\hat{h},1)$. ($\eta$ is taken with respect to $H(X^\omega)$.) Define $\hat{k}_2$ as before. There is least $\ell_3 \not\in \{j_1,j_2,\cdots,j_{m-1},\ell_1,\ell_2\}$. Then $\{\pi_{\ell_2\ell_3}(\hat{k}_2\hat{k}_1\hat{h}(a_1),\cdots\pi_{\ell_2\ell_3}(\hat{k}_2\hat{k}_1\hat{h}(a_m))\}$ is an m-element subset of X^2. Continue this process.

Define $k = \cdots\cdot\hat{k}_j\hat{k}_{j-1}\cdots\hat{k}\hat{h}$. By Fort's lemma $k \in H(X^\omega)$. (Note that $\rho(\hat{k}_{n+1}\cdots\hat{k}_1\hat{h},\hat{k}_n\cdots\hat{k}_1\hat{h}) = \rho(\hat{k}_{n+1},1) < \eta(\hat{k}_n\cdots \hat{k}_1 h,1)$.) Also, by construction, $k(a_i)(j) \neq k(a_\ell)(j)$ for $i,\ell \leq m$, $j \in N$. Likewise there is some k' in $H(X^\omega)$ such that $k'(b_i)(j) \neq k'(b_\ell)(j)$ for $i,\ell \leq m$, $j \in N$. For each odd $j \in N$, there is some $p_j \in H(X^2)$ such that $p_j(k(a_i)(j), k(a_i)(j+1)) = (k'(b_i)(j), k'(b_i)(j+1))$ for each $i \leq m$. Then $p = p_1 \times p_2 \times \cdots \in H(X^\omega)$ and $p(k(a_i)) = k'(b_i)$ for each $i \leq m$, and $k'^{-1}pk(a_i) = b_i$ for each $i \leq m$. Thus, X^2 is m-homogeneous implies X^n is m-homogeneous for $m \geq 2$, $m \in N$ and $n \in N$ or $n = \omega$, $n \geq 2$.

Suppose X^2 is countable dense homogeneous. Then X^2 is strongly m-homogeneous for each m in N and thus X^n is strongly m-homogeneous for each $m \in N$, n in $N \cup \{\omega\} - \{1\}$. Then X^n is countable dense homogeneous [U1].

Suppose X^2 is representable. Then X^2 admits a non-identity stable homeomorphism, and thus, so does X^n for $n \in N - \{1\}$. Also, X^n is 2-homogeneous. Then, by a result in [P1], X^n is representable.

The following fact has been independently observed by Wayne Lewis:

Theorem 13. *Suppose that* X *and* Y *are continua such that* $X \times Y$ *is* 2-*homogeneous. Then there are* 2 *distinct non-constant continuous functions* f *and* g *from* X *to itself such that* f *is homotopic to* g. (In fact, if $\varepsilon > 0$, f and g can be found such that $\rho(f,1_X) < \varepsilon$, and $\rho(g,1_X) < \varepsilon$.) *Further, if* $X \times Y$ *is representable, then if* $x \in X$, u *is open in* X *such that* $x \in u$ *and if* $y \in u$, *there is a homotopy* $F: X \times [0,1] \to X$ *such that* $F_0 = 1_X$, $F_1(x) = y$, $F_t(z) = z$ *for* $(z,t) \in (X - u) \times [0,1]$.

Proof. Suppose that $X \times Y$ is 2-homogeneous and $\varepsilon > 0$. Suppose $(x,y) \in X \times Y$. Now $X \times Y$ is locally connected [U2] and Y is locally connected. Consider $N_{\varepsilon/4}(1_{X\times Y})(x,y)$. There is some y' in Y such that (1) $y' \neq y$, (2) there is an arc $A: [0,1] \to Y$ from y to y' ($A_o = y, A_1 = y'$) such that $(x,A_t) \in N_{\varepsilon/4}(1_{X\times Y})(x,y)$ for $t \in [0,1]$. Choose x' such that $x' \neq x$ and there is $h \in N_{\varepsilon/4}(1_{X\times Y})$ such that $h((x,y),(x,y')) = ((x,y),(x',y'))$.

Then $\pi_X h \restriction X \times A([0,1])$ is a map from $X \times A[0,1]$ to X. Define $k: X \times [0,1] \to X$ as follows: $k(x,t) = \pi_X h(x,A_t)$. Then k is a homotopy, $k_o(x) = x$, $k_1(x) = x'$, $d_X(k_t(z),z) < \varepsilon$ for $(z,t) \in X \times [0,1]$. (Define $d_{X\times Y}(a,b) = d_X(a_X,b_X) + d_Y(a_Y,b_Y)$ for $(a,b) \in X \times Y$, where d_X, d_Y are metrics on X,Y, respectively, compatible with the respective topologies. Also, define ρ on $H(X \times Y)$ with respect to $d_{X\times Y}$.)

Suppose $X \times Y$ is representable, $x \in X$ and u is open such that $x \in u$. Pick $y \in Y$. (Take d_X, d_Y, $d_{X\times Y}$, ρ as in the first part.) Suppose V(1) is open in X, V(2) is open in Y such that (1) $(x,y) \in V(1) \times V(2) \subseteq u \times Y$; and (2) if $(r,s) \in V(1) \times V(2)$, there is some $h \in H(X \times Y)$ such that $h(x,y) = (r,s)$ and $h(z) = z$ for $z \notin V(1) \times V(2)$. Suppose $y' \in V(2)$, $y' \neq y$, such that there is an arc $A: [0,1] \to Y$ from $y = A_o$ to $y' = A_1$ in V(2). Then suppose o is open in $X \times Y$ such that (1) $(x,y) \in o \subseteq V(1) \times V(2)$; (2) $(X \times \{y'\}) \cap \overline{o} = \emptyset$ and (3) if $(r,s) \in o$ there is some k in $H(X \times Y)$ such that $k(x,y) = (r,s)$ and $k(z) = z$ for $z \notin o$.

There are D(1) open in X, D(2) open in Y such that $(x,y) \in D(1) \times D(2) \subseteq o$. Suppose $x' \in D(1)$. There is a homeomorphism ℓ in $H(X \times Y)$ such that $\ell(x,y) = (x',y)$ and $\ell(z) = z$ for $z \notin o$. Then consider $\pi_X \ell | X \times A([0,1])$: $X \times A(([0,1]) \to X$. Define $\hat{\ell}: X \times [0,1] \to X$ by

$\hat{\ell}(z,t) = \pi_X \ell(z,A_t)$ for $(z,t) \in X \times [0,1]$. Then $\hat{\ell}$ is a homotopy with $\hat{\ell}_1 = 1_X$, $\hat{\ell}_0(x) = x'$ and $\hat{\ell}_t(z) = z$ for $(z,t) \in (X-u) \times [0,1]$. Thus the second part is proved (with names scrambled).

References

[A] F. D. Ancel, *A proof and some applications of a theorem of E. G. Effros* (preprint).

[B] A. Beck, *On invariant sets*, Annals of Math. 67 (1958), 99-103.

[BB] B. L. Brechner, *On the dimensions of certain spaces of homeomorphisms*, Trans. Amer. Math. Soc. 121 (1966), 516-548.

[BN] M. Brown, *Weak* n-*homogeneity implies weak* n-1-*homogeneity*, Proc. Amer. Math. Soc. 10 (1959), 655-657.

[BR] W. Barit and P. Renaud, *There are no uniquely homogeneous spaces*, Proc. Amer. Soc. 68 (1978), 385-386.

[BT] R. Bennett, *Countable dense homogeneous spaces*, Fund. Math. 74 (1972), 189-194.

[E] E. G. Effros, *Transformation groups and* c*-*algebras*, Annals of Math. 81 (1965), 38-55.

[F] M. K. Fort, Jr., *Homogeneity of infinite products of manifolds with boundary*, Pacific J. Math. 12 (1962), 879-884.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*, 2nd Ed., Die Grundlehren der Mathematischen Wissenschaften, Band 115, Springer-Verlag, Berlin, 1979.

[K] J. Keesling, *Locally compact full-homeomorphism groups are zero-dimensional*, Proc. Amer. Math. Soc. 390-396.

[KKT] K. Kuperberg, W. Kuperberg, and W. R. R. Transue, *On the 2-homogeneity of Cartesian products*, Fund. Math. 110 (1980), 131-134.

[KK] K. Kuratowski, *Topology I and II*, Academic Press, New York, 1968.

[L1] W. Lewis, *Stable homeomorphisms of the pseudo-arc*, Canadian J. Math. 31 (1979), 363-374.

[L2] ________, *Homeomorphism groups and homogeneous continua* (preprint).

[LP] ________ and J. K. Phelps, *Stable homeomorphisms, Galois spaces, and related properties in homogeneous continua* (to appear in Texas Topology Summer '80 Proceedings).

[M] A. M. Macbeath, *Groups of homeomorphisms of a simply connected space*, Annals of Math. 79 (1964), 473-488.

[MZ] D. Montgomery and L. Zippin, *Topological transformation groups*, Robert F. Krieger Publishing Co., Huntington, N.Y., 1974.

[P] J. K. Phelps, *A condition under which 2-homogeneity and representability are the same in continua* (to appear in Fund. Math.).

[R] J. T. Rogers, Jr., *Decompositions of homogeneous continua*, Pacific J. Math. (to appear).

[U1] G. S. Ungar, *Countable dense homogeneity and n-homogeneity*, Fund. Math. 99 (1978), 155-160.

[U2] ________, *On all kinds of homogeneous spaces*, Trans. Amer. Math. Soc. 121 (1975), 393-400.

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