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## COHERENT DISTANCE FUNCTIONS

by

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## COHERENT DISTANCE FUNCTIONS

S.D. Shore

### 0. Introduction

Coherent distance functions (or semimetrics) were introduced in 1918 by A. D. Pitcher and E. W. Chittenden (see [PC]) with the intent of generalizing Fréchet's recent work in the "theory of point sets" and the "theory of real-valued (continuous) functions." Fréchet's thesis results (see [ $F_1$ ]) had been developed in the framework of (what was to be called) metric spaces in which, so it seems, the least agreeable property was the triangle inequality. Several mathematicians at that time were looking for a "more natural" context in which to prove Fréchet's results or, perhaps, some interesting generalizations.

With topological hindsight the notion of Pitcher and Chittenden arises "naturally." First of all, a *distance function* for  $X$  assigns a nonnegative real-valued "distance" between any two points  $p$  and  $q$  of  $X$  such that the distance between  $p$  and  $q$  is zero iff  $p = q$ . The "topological focus" at that time was the properties of "closed sets" and their relationship to "limit points" (see [ $F_4$ ]); a closed set was one which included its limit points. A major concern, in the absence of the triangle inequality, was whether or not the "limit of a sequence of limit points was itself a limit point." Pitcher and Chittenden captured that "desirable property" by defining *coherent* distance functions to be those distance functions which had the additional property that,

if  $d(x_n, p) \rightarrow 0$  and  $d(x_n, y_n) \rightarrow 0$ , then  
 $d(y_n, p) \rightarrow 0$  also.

It is easy to show that any metric is a coherent distance function and that the converse fails. The important question (at that time) was whether, given a coherent distance function  $d$ , there is an "equivalent metric"  $d_1$  (that is, a metric  $d_1$  such that  $d(x_n, p) \rightarrow 0$  iff  $d_1(x_n, p) \rightarrow 0$ ). The question arose naturally since Chittenden had just solved (in [C]) a similar question raised by Fréchet by showing that the notions of "voisinage" and "écart" were equivalent. In turn, the question raised by Pitcher and Chittenden was solved nine years later when V. W. Niemytzki (in [N]) showed that, for any coherent distance function there is an equivalent metric; an interesting sidelight is that Niemytzki's proof only shows that an equivalent metric exists without showing how one may explicitly "compute" the necessary metric from the given distance function.

One might think that Niemytzki's result would have closed the chapter on coherent distance functions, just as Chittenden's result ended all serious considerations of the "voisinage." But, that is not the case! The defining property and usefulness of coherent distance functions continued to arise (often without reference to earlier work) throughout this "1900-1940 Era of Metrization," which one could claim had as its central theme "find an equivalent metric in the context that some more general distance function was known to exist." This era came to an end with (the war and) the summarizing work of P. H. Frink [Fr].

Of course, the emergence of paracompactness on one hand and of Moore spaces (or developments) on the other produced the useful covering techniques which provided the crucial results for "cracking the metrization problem" that had been raised in 1923 by Alexandroff and Urysohn (see [AU]). It seems clear, however, that the "flavor" of the results had changed since no one was trying to "construct" an equivalent metric any longer, but rather, one was attempting to show that a metric existed without attempting to explicitly construct it.

Strangely enough, the case for the coherent distance functions did not die in the 1930's. The 1970's ushered in a renewed interest with the work of Kenton [K], Martin  $([M_1], [M_2], [M_3])$  and Harley and Faulkner  $([H_1], [HF])$ . And, again, the thrust of these papers made use of the interesting and useful defining property of these distance functions for establishing metrization theorems.

The purpose of this paper is to summarize the important properties of coherent distance functions, to show the usefulness of its defining property, and to suggest that perhaps, coherent distance functions are topologically more natural than metrics. (After all, coherent distance functions are defined entirely in terms of the "limit concept" whereas metrics rely on the "geometric notion" of the triangle inequality.) In particular, we prove the related, known metrization theorems as well as some new ones; in addition, we relate coherent distance functions to later notions, especially to the notion of a development (or Moore space) and to the notion of "normal metric."

### 1. Admissibility

If  $d$  is a metric for  $X$ , then the set of spheres determined by  $d$  is a base for a topology for  $X$ ; we use the notation  $S_d(x, \varepsilon)$  for the sphere  $\{y \in X \mid d(x, y) < \varepsilon\}$  which is centered at  $x$  with radius  $\varepsilon$ . In this way each metric is "naturally associated" with a unique topology for  $X$ .

In the case that  $d$  is a distance function for  $X$ , the spheres need not be a base for a topology. (A distance function for  $X$  is a nonnegative, *symmetric*, real-valued function  $d: X \times X \rightarrow \mathbb{R}$  such that  $d(p, q) = 0$  iff  $p = q$ .) The question of what the "naturally associated" topology should be has a couple of answers.

Historically, as one might have expected, the approach is to introduce a "closure"; that is, for any distance function  $d$  and any nonempty subset  $A$  of  $X$ , define  $d(p, A) = \inf\{d(p, a) \mid a \in A\}$ , the *distance from  $p$  to  $A$* , and  $d\text{-cl}[A] = \{y \in X \mid d(y, A) = 0\}$ , the  *$d$ -closure of  $A$* . The "idea," of course, is to support the  $d$ -closure as a topological closure and to claim that there is a topology  $\mathcal{G}$  for  $X$  such that  $d\text{-cl}[A] = \text{cl}_{\mathcal{G}}[A]$ , the closure of  $A$  in the topological space  $(X, \mathcal{G})$ . However, the  $d$ -closure need not be a topological closure (see [A] or [BH], for example); indeed, as the work of Pitcher and Chittenden might have suggested, the  $d\text{-cl}[d\text{-cl}[A]]$  need not be the  $d\text{-cl}[A]$ !

When there is a topology  $\mathcal{G}$  for  $X$  such that  $\text{cl}_{\mathcal{G}}[A] = d\text{-cl}[A]$ , then  $d$  is called an *admissible semimetric for  $\mathcal{G}$*  and  $(X, \mathcal{G})$  is said to be *semimetrizable*. (We also say, in this context, that  $(X, \mathcal{G})$  *admits the semimetric  $d$* .)

It is easy to show that  $d$  is admissible for  $(X, \mathcal{G})$  if and only if, for each  $p \in X$ , the set  $\{S_d(p, \varepsilon) \mid \varepsilon > 0\}$  of spheres centered at  $p$  is a neighborhood base for  $p$  in  $(X, \mathcal{G})$ . Hence, if  $d$  is an admissible semimetric for  $(X, \mathcal{G})$  whose spheres are open, then the set of spheres is a base for  $\mathcal{G}$ ; the converse need not hold. Heath's example in [H<sub>2</sub>] shows that a topological space  $(X, \mathcal{G})$  may be semimetrizable and still admit no semimetric whose set of spheres is a base for  $\mathcal{G}$ :

A semimetrizable space is always a first countable,  $T_1$ -space.

Arhangel'skiĭ [A] has recommended a "more naturally associated" topology to avoid the question of whether or not a distance function generates a topology. For any distance function  $d$  for  $X$ , consider the topology  $\mathcal{G}$  which consists of all sets  $G$  such that, if  $p \in G$ , then  $S_d(p, \varepsilon) \subseteq G$  for some  $\varepsilon$ ; this topology is called the *symmetric* topology for  $X$  generated by  $d$ . In this case we will say that  $d$  is an *admissible symmetric* for  $\mathcal{G}$  and that  $(X, \mathcal{G})$  is *symmetrizable*.

A symmetrizable space is always a  $T_1$ -space.

It is easy to see that any semimetrizable space is necessarily symmetrizable; indeed, if  $d$  is an admissible semimetric for  $\mathcal{G}$ , then  $d$  is an admissible symmetric. The converse may fail (see [A] or [BH]) even though a first countable, Hausdorff symmetrizable space is semimetrizable. (For a more complete comparison of semimetrizable and symmetrizable the reader is referred to the paper of Harley and Stephenson; see [HS].)

In the case that  $d$  is a coherent distance function, the situation is much simpler because there is *always* a Hausdorff topology  $\mathcal{G}$  for  $X$  such that  $d$  is an admissible semimetric for  $\mathcal{G}$ . This is what Pitcher and Chittenden had intended! Still, the set of spheres generated by  $d$  need not be a base for  $\mathcal{G}$ ; worse yet, perhaps, convergent sequences need not be Cauchy. Pitcher and Chittenden "solved" this latter problem by showing that, if  $d$  is a coherent distance function, then there is always an equivalent, coherent distance function  $d_1$  such that convergent sequences are Cauchy; namely, consider

$$d_1(x, y) = \inf\{d(x, z) + d(z, y) \mid z \in X\}.$$

(It is interesting to compare this with the metric which Frink constructed in [Fr] in her solution of the metrization problem.) It is easy to show that a distance function  $d$  has the property that its convergent sequences are Cauchy iff

when  $d(x_n, p) \rightarrow 0$  and  $d(y_n, p) \rightarrow 0$ ,  
 then  $d(x_n, y_n) \rightarrow 0$ .

This is the condition which was introduced by Pitcher and Chittenden in [PC]; in recent years a distance function with this property has been called *developable* since a semimetrizable space  $(X, \mathcal{G})$  admits such a semimetric if, and only if, there is a development for  $(X, \mathcal{G})$  in the sense of R. L. Moore (see section 3).

## 2. Metrization

It is a "standard exercise" in the study of metric spaces to show that any metric is uniformly continuous.

Hence, if  $d$  is a metric, then  $d$  is *1-uniformly continuous* (i.e., for each  $p \in X$ ,  $d_p: X \rightarrow \mathbf{R}$  is uniformly continuous, where  $d_p(x) = d(p, x)$ ). In a similar way one may define the notion of a *1-continuous* distance function. Of course, any metric is 1-continuous; indeed, if  $d$  is a metric, then  $d_A: X \rightarrow \mathbf{R}$  is continuous for any set  $A$ , where  $d_A(x) = d(x, A)$ . Recall that the functions  $d_A$  can play an important role in showing that any metrizable space is normal. Whether these functions  $d_A$  and these notions of 1-continuous and 1-uniformly continuous will be useful in the study of more general distance functions is discussed in the following Lemma and Theorem.

2.1. *Lemma. Consider the following for any admissible symmetric  $d$  for  $\mathcal{G}$ .*

- (1)  $d$  is a metric;
- (2)  $d$  is 1-uniformly continuous;
- (3)  $d_F: X \rightarrow \mathbf{R}$  is continuous for any nonempty closed set  $F$ ;

(4.1)  $d$  is coherent;

(4.2)  $d[K, F] > 0$  for any closed set  $F$  and disjoint compact set  $K$ , where  $d[K, F] = \inf\{d(x, F) \mid x \in K\}$ ;

(4.3) if  $p$  is not in the closed set  $F$ , then  $S_d(p, \delta) \cap S_d[F, \delta] = \emptyset$  for some  $\delta$ , where  $S_d[F, \delta] = \{x \mid d(x, F) < \delta\}$ .

Then, (1) implies (2) and (3), either of which implies (4.1) ... but, not conversely. Furthermore, (4.1), (4.2) and (4.3) are all equivalent.

2.2. *Proof and Remarks.* We have already remarked that (1) implies (2) and (3). It is not difficult to show



that (2) implies (4.1) and that (3) implies (4.2). Later we note that (4.2) implies (4.1) so that any distance function with either property (2) or (3) is coherent. The converses fail since a coherent distance function need not be 1-continuous (indeed, as Pitcher and Chittenden implicitly showed [PC], its spheres need not be open).

(4.1) *implies* (4.3). Suppose that  $d$  is coherent and that  $p \notin F$ , a closed set. There is  $\varepsilon_0$  such that  $S_d(p, \varepsilon_0) \cap F = \emptyset$ . Hence, there is  $\delta$  such that, if  $d(p, x) < \delta$  and  $d(x, y) < \delta$ , then  $d(p, y) < \varepsilon_0$ . (This is Niemytzki's "local écart" version of the coherent property!) It follows immediately that  $S_d(p, \delta) \cap S_d[F, \delta] = \emptyset$ .

(4.3) *implies* (4.2). Suppose that  $d$  satisfies (4.3); equivalently, if  $d(a_n, p) \rightarrow 0$  and  $d(a_n, b_n) \rightarrow 0$  for any sequence  $\langle b_n \rangle$  in the closed set  $F$ , then  $p \in F$ .

Hence, if  $K$  is a compact set and  $F$  is a closed set such that  $d[K, F] = 0$ , then we may choose  $a_n \in K$  and  $b_n \in F$  such that  $d(a_n, b_n) \rightarrow 0$ . But, since  $K$  is compact (or, equivalently, sequentially compact for symmetrizable spaces), there is a subsequence of  $\langle a_n \rangle$  which converges to a point in  $K$ . Hence, there are subsequences  $\langle a_{k_n} \rangle$  and  $\langle b_{k_n} \rangle$  such that  $d(a_{k_n}, b_{k_n}) \rightarrow 0$  and  $d(a_{k_n}, p) \rightarrow 0$  for some  $p \in K$ . From our supposition  $p \in F$  as well so that  $K$  and  $F$  are not disjoint.

This establishes the desired implication.

(4.2) *implies* (4.1). Suppose that  $d$  satisfies (4.2). Then, note that if  $x_n \rightarrow p$ , then  $d(x_n, p) \rightarrow 0$ . (Otherwise, there is a sequence  $\langle x_n \rangle$  such that  $x_n \rightarrow p$  and  $d(x_n, p) \geq \varepsilon$

for each  $n$ . If  $F = \{x_n \mid n \in \mathbb{N}\}$ , then  $d(p, F) > 0$ ; moreover, for any other  $q$ , not in  $F$ ,  $d(q, F) > 0$  since  $d[F \cup \{p\}, \{q\}] > 0$  because  $d$  satisfies (4.2). We conclude that  $F$  is closed which contradicts that  $x_n \rightarrow p$  and  $p \notin F$ .)

Now, claim that  $d$  is coherent. Otherwise, obtain  $x_n, y_n$  and  $p$  such that  $x_n \rightarrow p$  and  $d(x_n, y_n) \rightarrow 0$ , but  $y_n \rightarrow p$  fails. It follows that there are subsequences of  $\langle x_n \rangle$  and  $\langle y_n \rangle$  and an open set  $G$  such that  $p \in G$  and  $x_{k_n} \in G$  and  $y_{k_n} \notin G$  for each  $n$ . Thus, the closed set  $X \setminus G$  and the compact set  $\{p\} \cup \{x_{k_n} \mid n \in \mathbb{N}\}$  violate the property (4.2) which  $d$  satisfies.

2.3. *Remark.* It follows that, if  $d$  is a distance function for  $X$  with any of the properties of Lemma 2.1, then  $d$  is a *coherent* distance function. As such,  $d$  is necessarily an admissible semimetric for a first countable Hausdorff topology for  $X$ . Furthermore, according to Niemytzki [N], there is a metric for  $X$  which has the same convergent sequences ... so that this topology is, in fact, metrizable. Niemytzki's Result in concert with Lemma 2.1 establishes the following theorem.

2.4. *Metrization Theorem.* The following are equivalent:

- (1)  $(X, \mathcal{G})$  is metrizable.
- (2)  $(X, \mathcal{G})$  is 1-uniformly continuously symmetrizable.
- (3) There is an admissible symmetric  $d$  for  $\mathcal{G}$  such that  $d_F: X \rightarrow \mathbb{R}$  is continuous for any nonempty closed set  $F$ .
- (4) There is an admissible, coherent symmetric for  $\mathcal{G}$ .

(5) *There is an admissible symmetric  $d$  for  $\mathcal{G}$  such that  $d[K, F] > 0$  for any closed set  $F$  and disjoint compact set  $K$ .*

(6) *There is an admissible symmetric  $d$  for  $\mathcal{G}$  such that, when  $p$  is not in the closed set  $F$ , then  $S_d(p, \delta) \cap S_d[F, \delta] = \emptyset$  for some  $\delta$ .*

2.5. *Credits and Final Remark.* The equivalence of (1) through (6) was proved in [K] for the "case" of semi-metrics. That (1) and (4) are equivalent was, of course, the result of Niemytzki [N]. Martin in [M<sub>2</sub>] has a proof for the equivalence of (4) and (5) which he notes generalizes an earlier result (for Hausdorff spaces) of Arhangel'skiĭ [A]. The equivalence of (5) and (6) is proved in [HF].

We have noted that, in the case that  $d$  is a coherent distance function, there is an equivalent metric; however, it seems to be difficult to show how it may be "computed" from  $d$ . The "obvious choice" (in view of Frink's result [Fr]) is

$$d^*(p, q) = \inf \{ d(p, x_1) + \cdots + d(x_n, q) \mid \text{finite sets } x_1, \dots, x_n \text{ in } X \}.$$

There is no difficulty in showing that, if  $d^*$  is equivalent to the distance function  $d$ , then  $d$  is necessarily coherent. However, for an arbitrary coherent distance function  $d$ , the "computed"  $d^*$  need not be equivalent (in fact, it need not even be a metric, although it is always a continuous pseudo-metric).

### 3. Relationship to Covering Properties

Because of the importance of spaces which have a development (for example, Moore spaces) and because this

is intended to be a somewhat historical commentary on the role played by coherent distance functions, we ought to relate our theorems about distance functions (especially the metrization results) to the results of R. L. Moore and "his school."

Moore's idea that a "development" determines a topology originated in print at about the same time that Pitcher and Chittenden were introducing the notion of coherent distance function. A *development* for a topology  $\mathcal{G}$  is a sequence  $\langle \mathcal{U}_n \rangle$  of open covers such that, for each point  $p$  in  $X$ ,  $\{\text{st}(p, \mathcal{U}_n) \mid n \in \mathbf{N}\}$  is a local base for  $p$  in  $(X, \mathcal{G})$ , where  $\text{st}(p, \mathcal{U}_n) = \bigcup \{G \mid p \in G \in \mathcal{U}_n\}$ . Similarly, we define  $\text{st}[A, \mathcal{U}_n] = \bigcup \{\text{st}(a, \mathcal{U}_n) \mid a \in A\}$ , for any nonempty subset  $A$  of  $X$ .

Certainly it is clear that a development is intended to generalize for *developmentable spaces* the role played by spheres in metrizable spaces. Hence, it is not surprising that, if  $d$  is an admissible symmetric for  $\mathcal{G}$ , then we might want to consider

$$\mathcal{S}_n = \{\text{int}_{\mathcal{G}} S_d(p, r) \mid p \in X \text{ and } r \leq 2^{-n}\}.$$

In this context one can show:

3.1. *Lemma. If  $d$  is an admissible, coherent symmetric for  $\mathcal{G}$ , then  $\langle \mathcal{S}_n \rangle$  is a development for  $(X, \mathcal{G})$  such that  $\{\text{st}[\text{st}(p, \mathcal{S}_n), \mathcal{S}_n] \mid n \in \mathbf{N}\}$  is a local base for  $p$  in  $(X, \mathcal{G})$ .*

3.2. *Lemma. If  $d$  is an admissible, coherent symmetric for  $\mathcal{G}$ , then  $\langle \mathcal{S}_n \rangle$  is a development for  $(X, \mathcal{G})$  such that, for any closed set  $F$  and disjoint compact set  $K$ , there is*

a natural number  $n$  such that  $\text{st}[K, \mathcal{Y}_n) \cap F = \emptyset$ .

In consideration of a converse, suppose that  $\langle \mathcal{Y}_n \rangle$  is a sequence of open covers of  $X$ . In [AU] (see, also, [H<sub>3</sub>]) Alexandroff and Urysohn have suggested  $d(x, y) = 2^{-n}$ , where  $n$  is the first natural number such that  $x \notin \text{st}(y, \mathcal{Y}_n)$ , if such a natural number exists; otherwise,  $d(x, y) = 0$ . In this context it is not difficult to show these lemmas:

3.3. Lemma.  $\langle \mathcal{Y}_n \rangle$  is a development for  $(X, \mathcal{G})$  iff  $d$  is an admissible semimetric for  $\mathcal{G}$ . Moreover, in this case the semimetric  $d$  is a developable semimetric whose set of spheres is a base for  $\mathcal{G}$ .

3.4. Lemma.  $\langle \mathcal{Y}_n \rangle$  is a development for  $(X, \mathcal{G})$  such that, for each  $p$  in  $X$ ,  $\{\text{st}[\text{st}(p, \mathcal{Y}_n), \mathcal{Y}_n) \mid n \in \mathbb{N}\}$  is a local base for  $p$  in  $(X, \mathcal{G})$  iff  $d$  is an admissible, coherent symmetric for  $\mathcal{G}$ .

3.5. Lemma. If  $\langle \mathcal{Y}_n \rangle$  is a development for  $(X, \mathcal{G})$  such that, for any closed set  $F$  and disjoint compact set  $K$ , there is a natural number  $n$  such that  $\text{st}[K, \mathcal{Y}_n) \cap F = \emptyset$ , then  $d$  is an admissible coherent semimetric for  $\mathcal{G}$ .

In light of these lemmas and the Metrization Theorem of the preceding section we might easily write proofs for the well known metrization theorems of Moore and Jones; see [Mo] and especially [J].

3.6. Moore's Metrization Theorem.  $(X, \mathcal{G})$  is metrizable if, and only if,  $(X, \mathcal{G})$  has a development  $\langle \mathcal{Y}_n \rangle$  such that,

for each  $p$  in  $X$ ,  $\{\text{st}[\text{st}(p, \mathcal{U}_n), \mathcal{U}_n] \mid n \in \mathbb{N}\}$  is a local base for  $p$  in  $(X, \mathcal{G})$ .

3.7. *Jones' Metrization Theorem.*  $(X, \mathcal{G})$  is metrizable if, and only if,  $(X, \mathcal{G})$  has a development such that, for any closed set  $F$  and disjoint compact set  $K$ ,  $\text{st}[K, \mathcal{U}_n] \cap F = \emptyset$  for some natural number  $n$ .

#### 4. An Application

When a distance function  $d$  for  $X$  is continuous (i.e., when  $d(x_n, p) \rightarrow 0$  and  $d(y_n, q) \rightarrow 0$  implies that  $d(x_n, y_n) \rightarrow d(p, q)$ ), then it is admissible for a topology for  $X$ . Hence, a topological space  $(X, \mathcal{G})$  is continuously semimetrizable if it admits a continuous symmetric (or, equivalently, semimetric).

We apply the ideas of section 2 to the problem of showing that the Isbell-Mrowka spaces  $\Psi_{\mathcal{R}}$  are not continuously semimetrizable. It is well known and easy to prove that they are nonmetrizable Moore spaces. Recall (see [GJ;5I]) that, for any maximal, infinite family  $\mathcal{R}$  of infinite, almost disjoint subset of  $\mathbb{N}$ ,  $\Psi_{\mathcal{R}}$  denotes the space  $\mathbb{N} \cup \mathcal{R}$  which has as its topology the one generated by the local bases:

$$\{p\}, \text{ if } p \in \mathbb{N}; \{U_n(p) \mid n \in \mathbb{N}\}, \text{ if } p \in \mathcal{R},$$

$$\text{where } U_n(p) = \{m \in p \mid m \geq n\} \cup \{p\}.$$

For any such  $\mathcal{R}$ , we establish the following theorem.

4.1. *Theorem.*  $\Psi_{\mathcal{R}}$  is not continuously semimetrizable.

*Proof* (by contradiction). Suppose that  $d$  is an admissible continuous semimetric for  $\Psi_{\mathcal{R}}$ . Since  $\Psi_{\mathcal{R}}$  is not

metrizable, this semimetric can not be coherent; hence, there are sequences  $\langle a_n \rangle$  and  $\langle b_n \rangle$ , as well as a point  $p$ , in  $\mathbf{N} \cup \mathcal{R}$  such that

$$d(a_n, p) \rightarrow 0 \text{ and } d(a_n, b_n) \rightarrow 0, \text{ but } d(b_n, p) \rightarrow 0 \text{ fails.}$$

It is immediate that  $p$  is not a natural number. Moreover, rather than complicate the notation, we assume that the  $b_n$ 's are all distinct and that, for each  $n$ ,  $a_n \in (\{p\} \cup p)$  while  $b_n \notin p$ ; otherwise, choose appropriate subsequences with the stated properties.

First, consider the case that infinitely many of the  $b_n$ 's are natural numbers. From the maximality of  $\mathcal{R}$  there is  $B \in \mathcal{R}$  such that  $B \cap \{b_n \mid n \in \mathbf{N}\}$  is infinite; hence, there is a subsequence  $\langle b_{k_n} \rangle$  such that  $d(b_{k_n}, B) \rightarrow 0$ . It follows that  $K = \{a_n \mid n \in \mathbf{N}\} \cup \{p\}$  and  $H = \{b_{k_n} \mid n \in \mathbf{N}\} \cup \{B\}$  are disjoint compact sets with  $d[K, H] = 0$ . On the other hand, it is easy to show that, if  $d$  is continuous (as we have supposed), then  $d[K, H] > 0$  for any disjoint compact subsets  $H$  and  $K$ . This contradiction establishes the impossibility of the first case.

Now, consider the alternative: at most finitely many of the  $b_n$ 's are natural numbers. We assume that no  $b_n$  is a natural number; otherwise, we may consider appropriate subsequences. It follows that, by using the 1-continuity of  $d$  and the almost disjoint property of  $p$  relative to each  $b_n$ , we may define (recursively) a sequence  $\langle c_n \rangle$  in  $\mathbf{N}$  such that the  $c_n$ 's are all distinct and that, for each  $n$ ,  $c_n \in b_n$  while  $c_n \notin p$  and, finally, that  $d(a_n, c_n) < d(a_n, b_n) + 2^{-n}$ . It remains to apply the argument of the

first case to  $\langle c_n \rangle$  to obtain a contradiction.

Since a contradiction arises in either case, the proof is established.

4.2. *Remark.* A proof similar to this is presented in [K].

## 5. Normal Symmetrics

An admissible symmetric  $d$  for  $(X, \mathcal{G})$  is *normal* iff  $d[A, B] > 0$  for any disjoint closed sets  $A$  and  $B$ .

Obviously, if compact subsets are closed, then normal symmetrics are necessarily coherent (see Lemma 2.1). In general, however, normal symmetrics need not be coherent; consider (see [M<sub>2</sub>] where this example is attributed to Harley) the natural numbers and the semimetric  $d$  with  $d(m, n) = \frac{1}{|m - n|}$  (for  $m \neq n$ ) which is admissible for the finite-complement topology.

5.1. *Theorem.* If  $d$  is an admissible, normal symmetric for  $(X, \mathcal{G})$ , whose compact subsets are closed, then the set of non-isolated points in  $(X, \mathcal{G})$  is compact.

*Proof.* Suppose that  $d$  is an admissible, normal symmetric for  $(X, \mathcal{G})$  and that the compact subsets of  $(X, \mathcal{G})$  are closed; from Lemma 2.1 it follows that, in addition,  $d$  is a coherent semimetric.

We claim that the set of non-isolated points is compact. Otherwise, we may obtain a sequence  $\langle a_n \rangle$  of non-isolated points with no convergent subsequences; thus,  $A = \{a_n | n \in \mathbb{N}\}$  is closed. Since each  $a_n$  is non-isolated, we may obtain a second sequence  $\langle b_n \rangle$  of non-isolated points,



not in  $A$ , such that  $d(a_n, b_n) \rightarrow 0$ . Let  $B = \{b_n \mid n \in \mathbb{N}\}$ . Since  $B$  is disjoint from  $A$  and  $d[A, B] = 0$ , we conclude that  $B$  is not closed; i.e., there is a subsequence  $\langle b_{k_n} \rangle$  of  $b_n$  that converges to some  $p$  in  $X$ . Thus we have that  $d(b_{k_n}, p) \rightarrow 0$  and  $d(a_{k_n}, b_{k_n}) \rightarrow 0$  so that, since  $d$  is coherent,  $d(a_{k_n}, p) \rightarrow 0$ . This contradicts that  $\langle a_n \rangle$  had no convergent subsequences.

5.2. *Remark* Mrowka (see [Mr]) has shown that a topological space admits a normal metric iff it is a metrizable space whose set of non-isolated points is compact.

If  $d$  is an admissible, normal symmetric for a Hausdorff space  $(X, \mathcal{G})$ , then, as we have just seen, the set of non-isolated points is compact. Moreover,  $d$  is an admissible coherent symmetric for  $\mathcal{G}$ . According to the Metrization Theorem 2.4,  $(X, \mathcal{G})$  is metrizable.

Thus, we have the following theorem.

5.3. *Theorem.* For any Hausdorff space  $(X, \mathcal{G})$  the following are equivalent:

- (i)  $(X, \mathcal{G})$  admits a normal symmetric;
- (ii)  $(X, \mathcal{G})$  admits a normal metric;
- (iii)  $(X, \mathcal{G})$  is a metrizable space whose set of non-isolated points is compact.

5.4. *Remark.* Related results have been obtained recently by Martin; see  $[M_4]$  where additional references are also given.

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