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A LOCALLY PATHWISE CONNECTED NOT PATH–DETERMINED FRECHET SPACE, OR A METHOD OF CONSTRUCTING EXAMPLES

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1. Introduction I: The Example

We say that a collection $\mathcal{D}$ of subsets of a space X DETERMINES (the topology of) X if for all $F \subseteq X$, if $F \cap D$ is closed in D for each $D \in \mathcal{D}$ then F is closed in X[(2)]. We call a space X *path-determined* if it is determined by the collection of all paths in X, where in our terminology a path is the continuous image of [0,1] (and not a continuous map $[0,1] \to X$). Obviously each path-determined space is SEQUENTIAL, i.e. is determined by the collection of (images of) convergent sequences.

Franklin and Smith-Thomas, [1], have considered the question of whether there are reasonable conditions that imply path-determinedness; since one needs a rich supply of paths near each point it is reasonable to concentrate on locally pathwise connected spaces. As the product of ω_1

(1) Research done while supported by an NSF Grant.

(2) Ernest Michael has suggested that the currently usual terminology "X has the weak topology with respect to $\mathcal{D}$" should be changed since "weak topology" means something totally different to a functional analyst (who would think that X has the smallest, not the largest, topology that gives each member of $\mathcal{D}$ its subspace topology). The terminology, suggested by [1], that we here propose does not seem to have confusing connotations. Also, it does not use the ugly "with respect to."

factors [0,1] shows, a locally pathwise connected space need not be sequential, leave alone path-determined, hence some additional condition is needed. The following is taken from [1].

Proposition. A first countable locally pathwise connected space is path-determined.

This is quite elementary, and one would like the result to be true with "first countable" weakened to "sequential," or at least to "Fréchet"(3). The main result of [1] is that there is a space, for mysterious reasons called HE, which is Hausdorff, sequential and locally pathwise connected but not path-determined. Since HE is neither Fréchet nor regular, Franklin and Smith-Thomas ask if the Proposition remains true if "first countable" is replaced by "Fréchet", [1, 4.2], or by "sequential and regular (or normal", [1, 4.4]. We show that this is not the case.

Example. There is a σ-compact cosmic(4) regular locally pathwise connected Fréchet space, for obvious reasons called SHE, that is not path-determined.

This leaves open the question of whether one can replace "first countable" by "Hausdorff countably compact sequential," [1, 4.4], or by "compact Hausdorff Fréchet" in the Proposition.

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(3) A space X is called FRÉCHET if for all $A \subseteq X$ and $x \in X$, if $x \in \overline{A}$ then there is a sequence in A that converges to x.

(4) A space is called COSMIC if it is the continuous image of a separable metrizable space.

2. Introduction II: The Method

The example is found with a method of constructing examples that I have frequently found useful. I don't claim that I invented this method, it is just something I, and undoubtedly many others, have gradually become aware of. One can state some abstract generalities about the method, and that we do below. In this paper we illustrate how the method works with the construction of an actual example: SHE is tailor made for this purpose.

The method is to start with a rough idea of what the example looks like, and to become more and more specific as one takes care of more and more details. Of course, you always keep things as simple as possible. The question is what to do if problems arise. You could simply discard your first approach, try something else, and see if this time you are lucky. However, if you feel that your approach is right (simplicity and naturalness are key indications it is), then you shouldn't give up too soon. You could try a random change, but this is almost as bad as giving up your approach. Instead you now must analyze the problem, and pin point exactly what causes the problem. This is undefinable but easy to explain: knowing you have a problem is like being able to check a proof step by step, and pin pointing the exact cause is like understanding the proof. Having found the exact cause you make the smallest and most natural change that eliminates it.

What do you do if you can't find the exact cause? Don't give up yet, it might be possible to find it by looking away from it via the following trick. Perhaps you are in a situation where CH could be useful. You carefully design a plan for using CH, in particular try to think of all features you will need for technical reasons, or that might be convenient. Here it helps tremendously if you have extensive experience with CH of course, otherwise you have no choice but to actually perform the construction. Now one of those innocent looking helpful features might "be" the exact cause you were looking for. Alternatively, if you had to do the CH construction, and are cleaning up your construction of irrelevant rubbish, you may see why CH wasn't needed. Even if all you get is an example that seems to use CH in an essential way you are closer to an honest example--if there is one: you have more confidence in your approach, and know that there is an example, not a theorem--if there is an honest result.

Of course the above discussion also applies to finding proofs of theorems.

3. Destroying Path-Determinedness, I

As observed in [1], the obvious necessary and sufficient condition for a sequential space X to be path-determined is

(*) for every countable $C \subseteq X$ that converges (self-explanatory) there is a path P in X such that $P \cap C$ is infinite.

Let $\mathbb{N}$ be the POSITIVE integers, and let $\mathbb{N} \cup \{\infty\}$ be its one-point compactification. Our rough idea of SHE is that SHE is built around $\mathbb{N} \cup \{\infty\}$, and that every path in SHE intersects $\mathbb{N}$ in a finite set. SHE is to be locally pathwise connected, so we wish to have an arc[5] from ∞ to k for each $k \in \mathbf{N}$. As a first attempt we let SHE have

$$\{\infty\} \cup \mathbf{N} \times (0,1) \cup \mathbf{N}$$

as underlying set, and intend to make sure that

$\{\infty\} \cup \{k\} \times (0,1) \cup \{k\}$ is an arc, for each $k \in \mathbf{N}$.

For each point but ∞ we choose the obvious (from the picture you should have made) neighborhood system. No path in SHE is allowed to intersect $\mathbf{N}$ in an infinite set, in particular SHE herself is not a path. So we want SHE to be noncompact. To this end we create an infinite closed discrete set, for which the natural choice is $D = \mathbf{N} \times \{\frac{1}{2}\}$. However, each neighborhood of $\mathbf{N}$ contains a point of $\mathbf{N}$, and D cuts a point of the arc we put between ∞ and k, for each $k \in \mathbf{N}$, so no matter what neighborhood system we give ∞ (subject to D being closed discrete), the resulting space is not locally pathwise connected.

4. Restoring Local Pathwise Connectedness

The exact cause that our first attempt to construct SHE produced a space that is not locally path-wise connected cannot lie in other points than ∞, but also cannot lie in ∞, since D has to be closed discrete. So the exact cause is that there is only one arc from ∞ to k, for each $k \in \mathbf{N}$. This is easily remedied: We change the underlying set of SHE into

$$\{\infty\} \cup \mathbf{N} \times \mathbf{N} \times (0,1) \cup \mathbf{N}$$

and intend to make sure that

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[5] Copy of $[0,1]$.

(1) $\{\infty\} \cup \{\langle k,n\rangle\} \times (0,1) \cup \{k\}$ is an arc, for all $k,n \in \mathbf{N}$.

This way we have infinitely many arcs from ∞ to k, for each $k \in \mathbf{N}$, and obviously

(2) if we cut a convex piece out of $\{\langle k,n\rangle\} \times (0,1)$ for only finitely many $n \in \mathbf{N}$, then what is left of $\{\infty\} \cup \{k\} \times \mathbf{N} \times (0,1) \cup \{k\}$ is pathwise connected, for $k \in \mathbf{N}$.

Before we continue we emphasize that it is important to make the correct diagnosis of what the exact cause of the problem is: The authors of [1] also looked at the first attempt described in §3 (personal communication), but did not formulate the exact cause the way I just did, and made what I call a random (though motivated change): they replaced $\mathbf{N} \cup \{\infty\}$ by something more complicated and built their example around that.

Let us make a simple choice for a neighborhood system for all points of SHE but ∞, and for part of the neighborhood system at ∞, which is compatible with (1), and ensures that SHE is T_1, and that SHE is locally pathwise connected, and first countable (hence Fréchet), and regular at each point but ∞. Then we have a pretty accurate picture of what SHE looks like.

There is one obvious choice: For a point $\langle k,n,x\rangle \in \mathbf{N} \times \mathbf{N} \times (0,1)$ a basic neighborhood is to have the form $\{\langle k,n\rangle\} \times (a,b)$, with $0 \leq a < x < b \leq 1$. For a point $k \in \mathbf{N}$ a basic neighborhood is to have the form

$\{k\} \times \mathbf{N} \times (0,a) \cup \{k\}$, with $a \in (0,1)$[6]. For $k \in \mathbf{N}$ and $b \in (0,1)$ define

$$A_{k,b} = [1,k] \times \mathbf{N} \times (0,b] \cup [1,k] \quad ([1,k] \text{ an interval in } \mathbf{N})$$

Then at least all sets SHE - $A_{k,b}$, with $k \in \mathbf{N}$ and $b \in (0,1)$ are to be neighborhoods of ∞.

Note that every choice of a neighborhood system at ∞, compatible with the other neighborhood systems, will make SHE cosmic and σ-compact.

It remains to construct the entire neighborhood system at ∞.

5. Destroying Path-Determinedness, II

Suppose we have decided upon a neighborhood system of SHE, but that there is a path P in SHE such that $P \cap \mathbf{N}$ is infinite. Then $\infty \in P$. Since a path (= Peano continuum) is arcwise connected it follows that for each $k \in P \cap \mathbf{N}$ there is an $f(k) \in \mathbf{N}$ such that $\{\langle k,f(k)\rangle\} \times (0,k) \subseteq P$. Hence

(3) SHE is not path-determined if for each infinite $D \subseteq \mathbf{N}$ and each function f: $D \to \mathbf{N}$ the set

$$\{\langle k,f(k),x\rangle : k \in D, \text{ and } x \in (0,1)\}$$

includes an infinite closed discrete subset of SHE.

Our first try to build in (3) is to imitate what we did in §3: if we cut out a closed segment of (0,1) centered at level 1/2 for at most finitely many arcs joining

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(6) If you don't care about having first countability at all points but ∞ you may find another choice more natural (by definition I need not tell you what that choice is).

∞ and k, for each $k \in \mathbf{N}$, then the set left over is to be a neighborhood of ∞. Formally,

$$\mathcal{B} = \{\text{SHE} - (A_{k,b} \cup (S \times [\tfrac{1}{2} - w, \tfrac{1}{2} + w])): k \in \mathbf{N},$$
$$\text{and } b \in (0,1), \text{ and } S \subseteq \mathbf{N} \times \mathbf{N}, \text{ and}$$
$$[\forall k \in \mathbf{N}[S \cap \{k\} \times \mathbf{N} \text{ is finite}], \text{ and}$$
$$w \in (0,\tfrac{1}{2})\}$$

is to be a neighborhood system at ∞. It is easy to see that this is well-defined, that SHE is regular at ∞, and that SHE is locally pathwise connected at ∞ because of (2), of §5, but that SHE is not path-determined because of (3) above.

However, SHE is not Fréchet at ∞: Clearly the set of all points at level 1/2, i.e. $\mathbf{N} \times \mathbf{N} \times \{\frac{1}{2}\}$, has ∞ as (unique) cluster point, but has no sequence that converges to ∞ since every sequence in $\mathbf{N} \times \mathbf{N} \times \{\frac{1}{2}\}$ has infinitely many terms in some $A_{k,b}$ or some $S \times [\frac{1}{2} - w, \frac{1}{2} + w]$. (Note that it does not help to consider only "small" S, i.e. replacing "$\forall k \in \mathbf{N}[S \cap \{k\} \times \mathbf{N}]$ is finite" by $\exists m \in \mathbf{N}\, \forall k \in \mathbf{N}[|S \cap \{k\} \times \mathbf{N}| \leq m]$" does not help).

6. Diagnosis with CH

It is too early to give up our approach, so the problem that our present choice of SHE is not Fréchet must be that $\mathcal{B}$ is the wrong neighborhood system. That is not an accurate enough diagnosis of the exact cause of our problem. If you don't see the exact cause, and I certainly didn't see it, the method now dictates to use an additional set theoretic axiom, preferably something simple like CH.

How can CH be useful? From (3), of §5, we see that in order to destroy path-determinedness we have to look at each (partial) function in

$$\Phi = \{f\text{: there is an infinite } D \subseteq \mathbf{N} \text{ such that } f \text{ is a function } D \to \mathbf{N}\}.$$

Clearly this set has cardinality only ¢. Also, in order to ensure that SHE is Fréchet at ∞ we do not have to look at all $2^{¢} > ¢$ subsets of SHE: Since SHE is going to be hereditarily separable no matter what we do at ∞ it suffices to look only at the ¢ many countable subsets. So we do the usual thing: our new neighborhood base at ∞ will be the union of an increasing ω_1-chain of countable neighborhood systems that approximate it. At stage α, if $\mathcal{A}$ is the α^{th} approximation, we first look at the α^{th} countable set, call it C. If $\mathcal{A}$ thinks $\infty \in \overline{C}$ pick a sequence in C that converges to ∞, else don't do anything. Next look at the α^{th} member of Φ, and in accordance with (3), of §5, create a (possibly new) closed discrete set. This probably requires enlarging $\mathcal{A}$, without destroying any of the previously chosen convergent sequences, which looks like a feasible task since $\mathcal{A}$ is countable and since there are only countably many sequences to worry about.[7]

Before actually doing the CH construction we must have an idea of how to enlarge a neighborhood system, or how to create a new closed discrete set, without destroying convergent sequences. It looks like a good idea to let each

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[7] It even seems reasonable that this can be done under MA, with some obvious changes. The importance of this is not that we would have a consistency result under weaker assumptions than CH, but that we get more confidence that CH can be eliminated altogether, for it can at least be weakened.

of our sequences "converge" to a level (i.e. a set of the form $\mathbb{N} \times \mathbb{N} \times \{z\}$, with $z \in (0,1)$), and to put our closed discrete sets in different levels: For then a closed discrete set at one level will hardly affect a sequence that "converges" to a different level, at least if we take the obvious precaution to give the segments we cut out around our closed discrete sets smaller and smaller widths (and not the same width, i.e. the parameter w in the definition of $\mathcal{B}$ in §5).

7. The AHA-Erlebnis

We now see exactly what is wrong with the neighborhood system $\mathcal{B}$ for ∞ we defined in §5: It takes care of all members of Φ at the same level, and uses segments of the same width. We obviously do not need CH to assign different levels to members of Φ.

(I should point out that seeing the exact cause of the problem is somewhat easier if you read how an example was found than if you do the work yourself: I did §5 mentally, and via drawings, and while I used (2) I only verbalized it until I saw the exact cause.)

8. Restoring Fréchet

There is one technical difficulty left to take care of: we plan to cut out sets of the form $\{\langle k, f(k), z\rangle: k \in dom(f), |z - \lambda| \leq k^{-1}\}$ (observe that the widths converge to 0) for $f \in \Phi$, defined in §7, and $\lambda \in (0,1)$, with different λ's for different f's. The problem is that you might cut out two disjoint convex pieces from some open arc

$\{\langle k,n\rangle\} \times (0,1)$, and then the resulting neighborhood of ∞ is not pathwise connected. We take care of this difficulty by considering not all of Φ but a sufficiently big chunk. The precise way we do this is a matter of technique.

Call $f,g \in \Phi$ ALMOST DISJOINT if $f \cap g$ is finite, or, equivalently, if

$$\{k \in dom(f) \cap dom(g)\colon f(k) = g(k)\} \text{ is finite,}$$

and let M be a maximal (with respect to inclusion) pairwise almost disjoint subset of Φ. Let $\lambda\colon M \to [2/5,3/5]$[8] be an injection, and for $f \in M$ and $\varepsilon \in (0,1/5)$[8] define

$$B_{f,\varepsilon} = \{\langle k,f(k),z\rangle\colon k \in dom(f), \text{ and } |z - \lambda(f)| \le ((1/5) - \varepsilon)/k^{(8)}\}.$$

With $A_{k,b}$ as in §4 now let

$$\mathcal{B}' = \{\text{SHE} - (A_{k,b} \cup \bigcup_{f\in F} B_{f,\varepsilon})\colon k \in \mathbf{N}, \text{ and } b \in (0,1), \text{ and } F \text{ a finite subset of } M, \text{ and } \varepsilon \in (0,1/5)\}$$

be a neighborhood base at ∞. We omit the straightforward verification that SHE is well-defined, that SHE is regular at ∞, that (1) of §4, holds, and, of course, that $\mathbf{N}$ converges to ∞. Also, we already know from §4 that SHE is cosmic and σ-compact.

We check that SHE is locally pathwise connected at ∞. Given a finite $F \subseteq M$ there is $n \in \mathbf{N}$ such that for every two distinct $f,g \in F$, and for each $k \in dom(f) \cap dom(g)$, if $k > n$ then $f(k) \ne g(k)$. It now easily follows from (2),

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[8] We use 1/5, 2/5, 3/5 only for technical convenience: it ensures that $C_{f,\varepsilon} \subseteq \mathbf{N} \times \mathbf{N} \times (0,1)$, and will also be used in the proof that SHE is locally pathwise connected at ∞.

of §3, that SHE $- (A_{k,b} \cup \bigcup_{f\in F} B_{f,\varepsilon})$ is path-wise connected for $k > n$ and $b \in (4/5,1)$ and $\varepsilon \in (0,1/5)$.

We check that SHE is not path-determined. For each $f \in M$ the set $\{\langle k,f(k),\lambda(f)\rangle: k \in dom(f)\}$ clearly is closed discrete in SHE. Since M is maximal pairwise almost disjoint the conclusion now follows from (3) of §5.

We check that SHE is Fréchet at ∞. Consider any $L \subseteq$ SHE with $\infty \in \overline{L}$. We have to find a sequence in L that converges to ∞. Since $\mathbb{N}$ converges to ∞ we may assume that $L \subseteq \mathbb{N} \times \mathbb{N} \times (0,1)$. If there is $k \in \mathbb{N}$ such that $\infty \in \overline{L \cap \{k\} \times \mathbb{N} \times (0,1)}$ then there is a sequence in L that converges to ∞ since the subspace $\{\infty\} \cup \{k\} \times \mathbb{N} \times (0,1)$ of SHE is first countable at ∞. Hence we may assume that

(*) $\forall k \in \mathbb{N}[\infty \in \overline{L \cap [k,\infty) \times \mathbb{N} \times (0,1)}]$ ($[k,\infty)$ is an interval in $\mathbb{N}$).

For $i \in \{1,2,3\}$ let π_i be the projection of $\mathbb{N} \times \mathbb{N} \times (0,1)$ onto the i^{th} factor. Define the set P of potential sequences, and the cluster set K of L by

$P = \{s$: s is a sequence in L such that $\pi_1 \circ s$ is strictly increasing$\}$

$K = \{z \in [0,1]: \exists s \in P[\pi_3 \circ s \text{ clusters at } z]\} =$

$K = \{z \in [0,1]: \exists s \in P[\pi_3 \circ s \text{ converges to } z]\}$.

We wish to find $s \in P$ that converges to ∞. We make the following crucial observation, which shows that it was a good idea to let the widths of the segments that constitute $B_{f,\varepsilon}$'s get smaller and smaller:

(†) for all $s \in P$ and $f \in M$ and $\varepsilon \in (0,1/5)$, if $\lambda(f)$ if not a cluster point of $\pi_3 \circ s$ then $B_{f,\varepsilon}$ contains only finitely many terms of s.

Trivially we also have

(††) for all $s \in P$ and $k \in \mathbf{N}$ and $b \in (0,1)$ only finitely many terms of s belong to $A_{k,b}$.

From this it follows that we can handle the following two cases:

CASE 1. $K - \vec{\lambda}M$ is non-empty.

Take any $s \in P$ such that $\pi_3 \circ s$ converges to a point not in $\vec{\lambda}M$. Then s converges to ∞ by (†) and (††).

CASE 2. K is countable.

From (†) we see that the collection

$$\mathcal{A} = \{\text{SHE} - (A_{k,b} \cup \cup_{f \in F} B_{f,\varepsilon}) : k \in \mathbf{N} \text{ and } b \in \{1 - 2^{-n} : n \in \mathbf{N}\}, \text{ and } F \text{ is a finite subset of } M \text{ with } \vec{\lambda}F \subseteq K \text{ and } \varepsilon \in \{2^{-n} : n \in \mathbf{N}\}\}$$

sufficiently approximates $\mathcal{B}'$ in the sense that an element of P converges to ∞ in SHE iff it converges to ∞ if we give $\mathcal{A}$ as neighborhood base to ∞. Since $\mathcal{A}$ is countable we conclude from (††) that there is an $s \in P$ that converges to ∞. (Note the similarity with the CH construction!)

The remaining case obviously is too difficult to handle, so we make sure that it does not occur: Since K is a compact subset of [0,1] if suffices to require about $\lambda: M \to [2/5,3/5]$ that $\vec{\lambda}M$ does not have any uncountable compact subset. Such a λ can be found since $|M| \leq |\Phi| = \mathfrak{c}$, and since it is well known that [2/5,3/5] has a subset of cardinality $\mathfrak{c}$ which has no uncountable compact subsets, cf. [2, p. 422]. (It is not uncommon for such sets to be useful to avoid CH.)

Remark. Let ${}^{\mathbf{N}}\mathbf{N}$ denote the set of all functions $\mathbf{N} \to \mathbf{N}$. It is not hard to construct a subset of ${}^{\mathbf{N}}\mathbf{N}$ that is maximal almost disjoint in ${}^{\mathbf{N}}\mathbf{N}$ but not in Φ. Using CH, or MA, one can easily construct a subset of ${}^{\mathbf{N}}\mathbf{N}$ that is maximal disjoint in Φ. I have not seriously tried to construct such a set in ZFC.

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