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by

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ANOTHER CLASS OF λ—CONNECTED PRODUCTS

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Dedicated to the Memory of

Professor Patrick Henry Doyle

A compact metric continuum (hereinafter, simply "continuum") is of *type* λ [3, p. 262] if and only if it is irreducible between two points and contains no indecomposable subcontinuum with nonempty interior, or, equivalently, if and only if it admits a decomposition onto a closed interval whose elements are nowhere dense continua. A nondegenerate continuum is *λ-connected* [2] if and only if each two distinct points in it have a continuum of type λ irreducible between them. This is a delicate property since there exist continua of type λ which are not λ-connected. The two most obvious classes of λ-connected continua are the class of arcwise connected continua and the class of hereditarily decomposable continua. C. L. Hagopian [1] has proven the surprising result that the product of any two nondegenerate hereditarily indecomposable continua is λ-connected and has asked whether every product of nondegenerate continua is λ-connected. Hereditarily indecomposable continua are very far from being λ-connected, as they contain no subcontinua of type λ, so that Hagopian's result should take care of one of the most difficult cases of his problem. I would like to thank Hagopian for a brief

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but very helpful conversation on this problem at the Virginia Tech Topology Conference in March, 1981. At this time, Hagopian gave an argument that the product of two continua is λ-connected if each arc component of each factor is dense in the factor. Herein another special case of the problem is solved.

The letter d with the particular space subscripted will denote a metric; the diameter of a set is denoted dia(·). C denotes the usual Cantor ternary set.

Theorem. If X *and* Y *are* (compact, metric) *continua and* X *is* λ*-connected, then so is* $X \times Y$.

The proof of this theorem depends upon two lemmas.

Lemma 1. *Suppose* S *and* X *are continua and* $f: S \to X$ *is a continuous map. Suppose* $a, b \in S$ *and* X *is irreducible between* f(a) *and* f(b). *Suppose there is a dense* $R \subseteq S$ *such that* $f|R$ *is one-to-one and* $R = f^{-1}(f(R))$. *Then* S *is irreducible from* a *to* b, *and if* X *is of type* λ, *so is* S.

Proof. First observe that if E is any proper closed subset of S, then $f(E) \neq X$, since by the conditions on R, f(E) cannot contain any point of the nonempty set $f(R \cap (S-E))$. Thus, if E contains both a and b, f(E) cannot be connected, by irreducibility of X from f(a) to f(b). If f(E) is not connected, neither is E, proving that S is irreducible from a to b. A similar argument shows that if E is a nowhere dense closed subset of S, then f(E) is nowhere dense in X.

Suppose X is of type λ and suppose $J \subseteq S$ is an indecomposable continuum with nonempty interior. (Referring to Figure 1 from time to time may be helpful here.) Let A and B be subcontinua of S irreducible from a to J and from b to J, respectively. (Of course, either A or B could be degenerate.) Now, $A \cup B \neq S$, so that $f(A \cup B) \neq X$. By a standard irreducibility argument, $X - f(A \cup B)$ is connected; thus $Cl(X - f(A \cup B))$ is a subcontinuum of X with nonempty interior, and so is decomposable. Let K, L be proper subcontinua of $Cl(X - f(A \cup B))$ whose union is $Cl(X - f(A \cup B))$. Assume without loss of generality that $K \cap f(A) \neq \emptyset$. Then, $K \cap f(B) = \emptyset$; $L \cap f(A) = \emptyset$; and $L \cap f(B) \neq \emptyset$. Furthermore, each of $X - (f(A \cup B) \cup K)$ and $X - (f(A \cup B) \cup L)$ is a nonempty open set. Let $p \in X - (f(A \cup B) \cup K)$, and let $\varepsilon > 0$ be such that the ε-neighborhood of p is contained in $X - (f(A \cup B) \cup K)$. Let $\delta > 0$ be such that if $x,y \in S$ and $d_S(x,y) < \delta$, then $d_X(f(x),f(y)) < \varepsilon$. Let $s \in J \cap A$. Since J is indecomposable, there exists a subcontinuum $W \subseteq J$ with $s \in W$ such that W is both nowhere dense in J and δ-dense in J. In particular, there exists $r \in W$ and $q \in f^{-1}(p)$ such that $d_S(r,q) < \delta$. Hence $d_X(f(r),p) < \varepsilon$, so that

$$f(r) \in X - (f(A \cup B) \cup K) \subseteq L.$$

Thus, f(W) meets both f(A) and L, so that $f(A) \cup f(W) \cup L \cup f(B) = X$, by irreducibility of X. Then f(W) must contain the nonempty open set $X - (f(A \cup B) \cup L)$, which is impossible since W is nowhere dense in J and hence in S. Therefore, S is of type λ if X is and the proof is complete.

Lemma 2. *Suppose* X *is a continuum of type* λ *irreducible from* α *to* β *while* Y *is an arbitrary continuum. Then there is a subcontinuum* $S \subseteq X \times Y$ *of type* λ *containing* $\{\alpha,\beta\} \times Y$ *and irreducible from any point of* $\{\alpha\} \times Y$ *to any point of* $\{\beta\} \times Y$.

Proof. Let $F: X \times Y \to X$ be the projection map and let $g: X \to [0,1]$ be a standard quotient mapping, as mentioned in the introduction, with $g(\alpha) = 0$, $g(\beta) = 1$, and each $g^{-1}(t)$ a layer of X. Since the layers of continuity form a dense G_δ in X [4, p. 202], g may be chosen so that $g^{-1}(C \cap (0,1))$ contains only layers of continuity. (A layer of continuity is a layer $g^{-1}(t)$ with the property that given any sequence $\langle t_n \rangle$ converging to t, $g^{-1}(t_n)$ converges to $g^{-1}(t)$.)

Let $\{(a_i,b_i)\}_{i=1}^{\infty}$ be a countable dense subset of $F^{-1}(g^{-1}(C))$ such that the sequence $\langle g \circ F(a_i,b_i) \rangle_{i=1}^{\infty}$, or $\langle g(a_i) \rangle_{i=1}^{\infty}$ contains the left-hand endpoint of each complementary interval of C in [0,1] exactly once. It is fairly straightforward to construct such a set, using the fact that $g^{-1}(C)$ contains only layers of continuity, save perhaps for $g^{-1}(0)$ and $g^{-1}(1)$. Let $g(a_i) = \hat{a}_i$ for each i, and let $\bar{a}_i$ denote the right-hand endpoint of the interval in $[0,1] - C$ of which $\hat{a}_i$ is the left endpoint. Thus $\langle (\hat{a}_i,\bar{a}_i) \rangle_{i=1}^{\infty}$ is an enumeration of the components of $[0,1] - C$.

Define

$$S = F^{-1}(g^{-1}(C)) \cup \left(\bigcup_{i=1}^{\infty} (g^{-1}[\hat{a}_i,\bar{a}_i] \times \{b_i\}\right),$$

and let $R = \bigcup_{i=1}^{\infty} (g^{-1}(\hat{a}_i,\bar{a}_i) \times \{b_i\})$. Then S is a union of a Cantor set of continua, $F^{-1}(g^{-1}(C))$, together with a countable collection of continua, $g^{-1}[\hat{a}_i,\bar{a}_i] \times \{b_i\}$, bridging the gaps represented by the complementary intervals of C. (A schematic picture for the special case X = Y = [0,1] is given in Figure 2.)

Note that $(g^{-1}\{\hat{a}_i,\bar{a}_i\}) \times \{b_i\}$ is contained in $F^{-1}(g^{-1}(C))$, since $\hat{a}_i,\bar{a}_i \in C$. R is dense in S, for suppose $U \times V$ is any basic open set in $X \times Y$ meeting S. If $U \times V \subseteq R$ we are done, so suppose $(U \times V) \cap F^{-1}(g^{-1}(C)) \neq \emptyset$. Then for some i, $(a_i,b_i) \in U \times V$, and since $a_i \in g^{-1}(\hat{a}_i)$ and $g^{-1}(\hat{\mathbf{a}}_i)$ is a layer of cohesion (actually a layer of continuity) of X, $U \cap g^{-1}(\hat{a}_i,\bar{a}_i) \neq \emptyset$. Since $b_i \in V$, $(U \times V) \cap (g^{-1}(\hat{a}_i,\bar{a}_i) \times \{b_i\}) \neq \emptyset$ as well, so that $(U \times V) \cap R \neq \emptyset$.

Let $f = F|S$. It is easy to see that $f|R$ is one-to-one and that $f^{-1}(f(R)) = R$; hence Lemma 1 tells us that S is irreducible from any point of $\{\alpha\} \times Y$ to any point of $\{\beta\} \times Y$ and is of type λ, completing the proof.

Proof of Theorem. Suppose X is a nondegenerate λ-connected continuum and Y is an arbitrary continuum. Let (a,y) and (b,v) be distinct points in $X \times Y$. If $a \neq b$, let $\hat{X}$ be a subcontinuum of X of type λ irreducible from a to b. Then by Lemma 2 there is a continuum S of type λ lying in $\hat{X} \times Y$ irreducible from (a,y) to (b,v). Thus we may assume that a = b and $y \neq v$. (The following construction is similar to one suggested to me by Hagopian.) Let $c \in X$, $c \neq a$ and let $\hat{X} \subseteq X$ be a continuum irreducible from a to c. Let U_n be the open neighborhood of y of radius $\frac{1}{n} \cdot d_Y(y,v)$. Let Y_n be the component of $Y - U_n$

containing v, and let $\hat{Y} = \mathrm{Cl}\left(\bigcup_{n=1}^{\infty} Y_n\right)$.

Now let $g: \hat{X} \to [0,1]$ be a continuous function whose point inverses are layers, with $g(a) = 0$ and $g(c) = 1$, [4, p. 199] and assume, with no loss of generality, that for each integer $n > 1$, $g^{-1}(1 - \frac{1}{n})$ is a layer of cohesion [4, p. 201]. Let $X_n = g^{-1}([1 - \frac{1}{n}, 1 - \frac{1}{n+1}])$. Choose $a_n \in g^{-1}(1 - \frac{1}{n})$ such that $a_1 = a$. Then X_n is irreducible from a_n to a_{n+1} for every n.

By Lemma 2, in each $X_n \times Y_n$ there is a continuum S_n of type λ irreducible from every point of $\{a_n\} \times Y_n$ to every point of $\{a_{n+1}\} \times Y_n$. Then, $S_0 = \bigcup_{n=1}^{\infty} S_n \cup (g^{-1}(1) \times \hat{Y})$ is a continuum of type λ which is irreducible from (a,v) to each point of $g^{-1}(1) \times \hat{Y}$, and which meets $X \times \{y\}$ only in some points of $g^{-1}(1) \times \hat{Y}$. Hence $S = S_0 \cup (\hat{X} \times \{y\})$ is a continuum of type λ irreducible from (a,y) to (a,v), and the theorem is proven. (This construction is schematically illustrated in Figure 3.)

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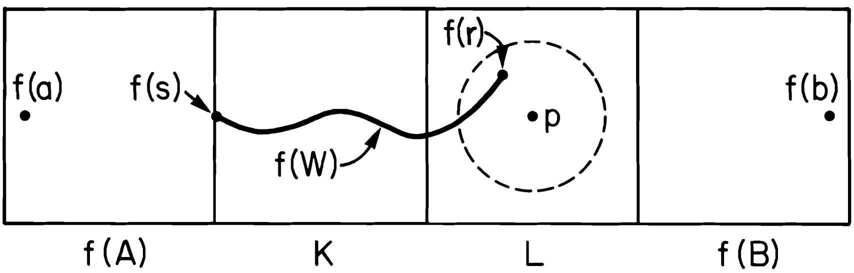

FIGURE 1: Schematic picture of X for last part of proof of Lemma 2.

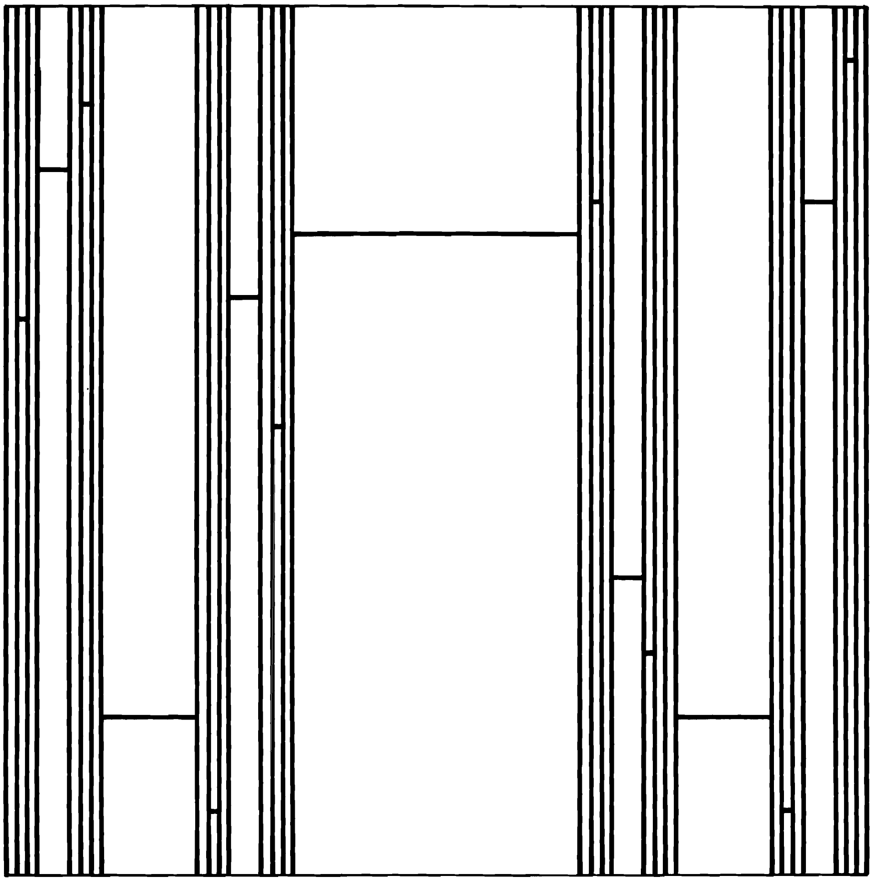

FIGURE 2: S, for the case X=Y= [0,1].

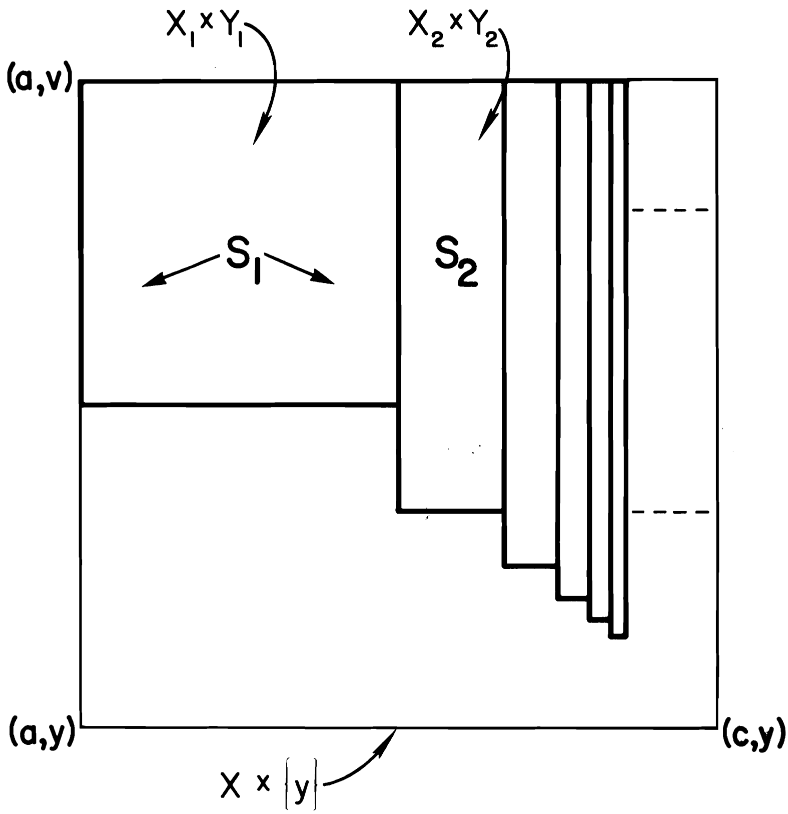

FIGURE 3: Each S_n is a subcontinuum of the "rectangle" $X_n \times Y_n$, irreducible from one side of it to the other.