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1. Introduction

While working on my Ph.D. Thesis, I began studying 2^X, the set of all nonempty closed subsets of a topological space X, and the various topologies that have been defined on 2^X. Since I was also studying the theory of nets and filters at the time, I decided to define a topology on 2^X using the nets of closed sets in 2^X. One way to do this is as follows:

limsup (A_d) = $\{x \in X |$ whenever U is an open set containing x, $U \cap A_d$ is nonempty for all d in a cofinal subset D_U of $D\}$.

liminf (A_d) = $\{x \in X |$ whenever U is an open set containing x, $U \cap A_d$ is nonempty for all $d \geq d_U$, where $d_U \in D\}$.

If (A_d) is a net such that limsup (A_d) = liminf (A_d) = A then we say that (A_d) is *topologically convergent* and we write lim (A_d) = A. If A is nonempty we write Lim (A_d) = A, using the capital "L" for Lim.

2. Definition of the Topology of Convergence

Since clearly,

(i) Every constant net is topologically convergent.

(ii) Every subnet of a topologically convergent net is topologically convergent and has the same limit.

we can define a closure operator cl: $P(2^X) \to P(2^X)$ as follows: For $a \in 2^X$,

$$\text{cl } \alpha = \{F \in 2^X \mid \text{there is a net } (A_d) \text{ in } \alpha \text{ with } \operatorname{Lim}(A_d) = F\}.$$

Since topological convergence satisfies (i) and (ii) above, the operator cl satisfies the first three Kuratowski closure axioms. [5] (note that in general cl(cl(α)) may not be equal to cl(α) so that we may have a set α with cl(α) $\subsetneq \bar{\alpha}$; see Kelley [4].) Thus we have defined a topology on 2^X, called the topology of convergence, where a subset $\alpha \subseteq 2^X$ is closed in this topology if and only if cl(α) = α.

3. A Comparison to Other Topologies on 2^X

The Vietoris topology on 2^X has as a subbase.

$$\mathcal{S} = \{\langle U \rangle, \langle X,V \rangle \mid U,V \text{ are open subsets of } X\}$$

where

$$\langle U \rangle = \{F \in 2^X \mid F \text{ is contained in } U\}$$

and

$$\langle X,V \rangle = \{F \in 2^X \mid F \cap V \text{ is nonempty}\}.$$

(See Vietoris [8] and Michael [6].)

Fell's topology has as a subbase

$$\mathcal{S} = \{\langle X-C \rangle, \langle X,U \rangle \mid C \text{ is a compact subset of } X \text{ and } U \text{ is an open subset of } X.\}$$

(See Fell [2].)

When X is a compact Hausdorff space we find that the topology of convergence is the Vietoris topology and when X is locally compact, the topology of convergence is Fell's topology. (See Frolik [3] and Mrowka [7].) In the non-locally compact case, we can prove the following:

Theorem. When X *is a* T_3*, non-locally compact space, the topology of convergence is strictly between Fell's* *topology and the topology of convergence.*

Proof. We wish to show the following:

$$\text{Fell's} \subsetneq \text{topology of convergence} \subsetneq \text{Vietoris}$$

The interesting part of the proof is showing that the containments are proper. We leave the remainder of the proof to the reader.

To show the second containment is proper: Let (x_d) be a net in X with no cluster point. Then for any $x \in X$ there is an open set U and $d_0 \in D$ such that $x \in U$, $x_d \notin U$ for all $d \geq d_0$. Define $(F_d)_{d \geq d_0}$ where $F_d = \{x_d, x\}$. Then $\text{Lim}\ (F_d) = \{x\}$ but (F_d) does not converge to $\{x\}$ in the Vietoris topology since $F_d \notin \langle U \rangle$ for all $d \geq d_0$.

To show the first containment is proper, we use a construction due to Mrowka [7]. Since X is not locally compact, there is $x_0 \in X$ with no neighborhood with compact closure. Let D_1 be the base of open sets at x_0 directed by set inclusion. Let $x_1 \neq x_0$ and $(y_k)_{k \in D_2}$ be a net with no cluster point. Let $D = D_1 \times D_2$ and let $\{\phi_i\}_{i=1,2}$ be the projection maps. Set $U_n = \phi_1(n)$ and $x_n = y_{\phi_2(n)}$ for each $n \in D$. Since $\overline{U}_n$ is not compact there is a net $(x_m^n)_{m \in E_n}$ in $\overline{U}_n$ with no cluster point. For $n \in D$, $m \in E_n$ set

$$A_{n_m} = \{x_1, x_n, x_m^n\}$$

Then for each n, $\text{Lim}\ (A_{n_m})_{m \in E_n} = \{x_1, x_n\} = A_n$ so that each (A_{n_m}) converges in Fell's topology to A_n. Also, $\text{Lim}\ (A_n)_{n \in D} = \{x_1\}$ and so $(A_n)_{n \in D}$ converges in Fell's topology to $\{x_1\}$.

Let $(A_{n_m})_{n\in D, m\in E_n}$ be the net ordered lexicographically. It can be shown that $x_0 \in \liminf (A_{n_m})_{n\in D, m\in E_n}$. Also, $(A_{n_m})_{n\in D, m\in E_n}$ has a subnet (A_s) that converges in Fell's topology to $\{x_1\}$. (See Kelley's [4] Theorem on Iterated Limits.)

Now (A_s) has a topologically convergent subnet (A_b), (see Chimenti [1]), say $\liminf (A_b) = \limsup (A_b) = A$, and since (A_b) is a subnet of $(A_{n_m})_{n\in D, m\in E_n}$ we have $x_0 \in A$. Thus (A_b) converges topologically to A, but converges in Fell's topology to $\{x_1\}$ where $A \neq \{x_1\}$.

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