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## A COMPARISON OF TOPOLOGIES FOR $2^X$

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## A COMPARISON OF TOPOLOGIES FOR $2^X$

**Barbara A. Flajnik**

### 1. Introduction

While working on my Ph.D. Thesis, I began studying  $2^X$ , the set of all nonempty closed subsets of a topological space  $X$ , and the various topologies that have been defined on  $2^X$ . Since I was also studying the theory of nets and filters at the time, I decided to define a topology on  $2^X$  using the nets of closed sets in  $2^X$ . One way to do this is as follows:

$$\text{limsup } (A_d) = \{x \in X \mid \text{whenever } U \text{ is an open set containing } x, U \cap A_d \text{ is nonempty for all } d \text{ in a cofinal subset } D_U \text{ of } D\}.$$

$$\text{liminf } (A_d) = \{x \in X \mid \text{whenever } U \text{ is an open set containing } x, U \cap A_d \text{ is nonempty for all } d \geq d_U, \text{ where } d_U \in D\}.$$

If  $(A_d)$  is a net such that  $\text{limsup } (A_d) = \text{liminf } (A_d) = A$  then we say that  $(A_d)$  is *topologically convergent* and we write  $\text{lim } (A_d) = A$ . If  $A$  is nonempty we write  $\text{Lim } (A_d) = A$ , using the capital "L" for Lim.

### 2. Definition of the Topology of Convergence

Since clearly,

(i) Every constant net is topologically convergent.

(ii) Every subnet of a topologically convergent net is topologically convergent and has the same limit.

we can define a closure operator  $\text{cl}: P(2^X) \rightarrow P(2^X)$  as follows: For  $a \in 2^X$ ,

$$\text{cl } a = \{F \in 2^X \mid \text{there is a net } (A_d) \text{ in } a \text{ with} \\ \text{Lim } (A_d) = F\}.$$

Since topological convergence satisfies (i) and (ii) above, the operator  $\text{cl}$  satisfies the first three Kuratowski closure axioms. [5] (note that in general  $\text{cl}(\text{cl}(a))$  may not be equal to  $\text{cl}(a)$  so that we may have a set  $a$  with  $\text{cl}(a) \subsetneq \bar{a}$ ; see Kelley [4].) Thus we have defined a topology on  $2^X$ , called the topology of convergence, where a subset  $a \subseteq 2^X$  is closed in this topology if and only if  $\text{cl}(a) = a$ .

### 3. A Comparison to Other Topologies on $2^X$

The Vietoris topology on  $2^X$  has as a subbase.

$$\mathcal{S} = \{\langle U \rangle, \langle X, V \rangle \mid U, V \text{ are open subsets of } X\}$$

where

$$\langle U \rangle = \{F \in 2^X \mid F \text{ is contained in } U\}$$

and

$$\langle X, V \rangle = \{F \in 2^X \mid F \cap V \text{ is nonempty}\}.$$

(See Vietoris [8] and Michael [6].)

*Fell's topology* has as a subbase

$$\mathcal{S} = \{\langle X-C \rangle, \langle X, U \rangle \mid C \text{ is a compact subset of } X \text{ and} \\ U \text{ is an open subset of } X.\}$$

(See Fell [2].)

When  $X$  is a compact Hausdorff space we find that the topology of convergence is the Vietoris topology and when  $X$  is locally compact, the topology of convergence is Fell's topology. (See Frolik [3] and Mrowka [7].) In the non-locally compact case, we can prove the following:

*Theorem.* *When  $X$  is a  $T_3$ , non-locally compact space, the topology of convergence is strictly between Fell's*

*topology and the topology of convergence.*

*Proof.* We wish to show the following:

Fell's  $\overset{c}{\neq}$  topology of convergence  $\overset{c}{\neq}$  Vietoris

The interesting part of the proof is showing that the containments are proper. We leave the remainder of the proof to the reader.

To show the second containment is proper: Let  $(x_d)$  be a net in  $X$  with no cluster point. Then for any  $x \in X$  there is an open set  $U$  and  $d_0 \in D$  such that  $x \in U$ ,  $x_d \notin U$  for all  $d \geq d_0$ . Define  $(F_d)_{d \geq d_0}$  where  $F_d = \{x_d, x\}$ . Then  $\text{Lim } (F_d) = \{x\}$  but  $(F_d)$  does not converge to  $\{x\}$  in the Vietoris topology since  $F_d \not\subseteq \langle U \rangle$  for all  $d \geq d_0$ .

To show the first containment is proper, we use a construction due to Mrowka [7]. Since  $X$  is not locally compact, there is  $x_0 \in X$  with no neighborhood with compact closure. Let  $D_1$  be the base of open sets at  $x_0$  directed by set inclusion. Let  $x_1 \neq x_0$  and  $(y_k)_{k \in D_2}$  be a net with no cluster point. Let  $D = D_1 \times D_2$  and let  $\{\phi_i\}_{i=1,2}$  be the projection maps. Set  $U_n = \phi_1(n)$  and  $x_n = y_{\phi_2(n)}$  for each  $n \in D$ . Since  $\bar{U}_n$  is not compact there is a net  $(x_m^n)_{m \in E_n}$  in  $\bar{U}_n$  with no cluster point. For  $n \in D$ ,  $m \in E_n$  set

$$A_{n,m} = \{x_1, x_n, x_m^n\}$$

Then for each  $n$ ,  $\text{Lim } (A_{n,m})_{m \in E_n} = \{x_1, x_n\} = A_n$  so that each  $(A_{n,m})$  converges in Fell's topology to  $A_n$ . Also,  $\text{Lim } (A_n)_{n \in D} = \{x_1\}$  and so  $(A_n)_{n \in D}$  converges in Fell's topology to  $\{x_1\}$ .

Let  $(A_{n_m})_{n \in D, m \in E_n}$  be the net ordered lexicographically. It can be shown that  $x_0 \in \liminf (A_{n_m})_{n \in D, m \in E_n}$ . Also,  $(A_{n_m})_{n \in D, m \in E_n}$  has a subnet  $(A_s)$  that converges in Fell's topology to  $\{x_1\}$ . (See Kelley's [4] Theorem on Iterated Limits.)

Now  $(A_s)$  has a topologically convergent subnet  $(A_b)$ , (see Chimenti [1]), say  $\liminf (A_b) = \limsup (A_b) = A$ , and since  $(A_b)$  is a subnet of  $(A_{n_m})_{n \in D, m \in E_n}$  we have  $x_0 \in A$ . Thus  $(A_b)$  converges topologically to  $A$ , but converges in Fell's topology to  $\{x_1\}$  where  $A \neq \{x_1\}$ .

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