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J. Stallings [9] asked the question: "If one considers $I = [0,1]$ embedded in $I \times I = I^2$ as $I \times 0$, can a connectivity function $I \to X$ be extended to a connectivity function $I^2 \to X$?" J. L. Cornette [3] and J. H. Roberts [8] gave negative answers to this question. A natural question arises: "In order for the extension $I^2 \to I$ of a connectivity function $I \to I$ to be a connectivity function, what is both necessary and sufficient?" Toward this end we defined the Cantor Intermediate Value Property (CIVP) which is given in definition 2. The relationship between the Cantor Intermediate Value Property and this question is unknown. However we conjecture that if the extension $I^2 \to I$ of a connectivity function $f: I \to I$ is a connectivity function, then f has the CIVP.

In this paper we give the relations between continuous functions, connectivity functions, Darboux functions, and functions having the CIVP.

Definition 1. A function $f: X \to Y$ between spaces X and Y is a connectivity function if and only if for each connected subset C of X, the graph of f restricted to C, denoted by $f|C$, is a connected subset of $X \times Y$. The function f is a Darboux function if $f(C)$ is connected for each connected set C.

Definition 2. A function $f: I \to I$ has the Cantor Intermediate Value Property (CIVP) if and only if for any Cantor set K in the interval $(f(x), f(y))$ the interval (x,y) or (y,x) contains a Cantor set C such that $f(C) \subset K$.

It is well known that there are Darboux functions which are not connectivity functions [1], [2]. Since the projection map is continuous, each connectivity function is a Darboux function. Also there are connectivity functions which are not continuous. However, every continuous function is a connectivity function. To complete the relations between these functions we (1) construct a connectivity function which does not have the CIVP, (2) construct a function which has the CIVP but is not a Darboux function, and (3) prove that if $f: I \to I$ is a continuous function, then f has the CIVP.

A Hamel basis for the real numbers is a set of numbers $a, b, c, \cdots$ such that if x is any number then x may be expressed uniquely in the form $\alpha a + \beta b + \gamma c + \cdots$ where $\alpha, \beta, \gamma, \cdots$ are rational numbers of which only a finite number are different from zero. We now construct a Hamel basis for the real numbers which is a subset of I and has cardinality c in each interval of positive length but contains no Cantor set.

As stated in [5], Burstin showed the existence of a Hamel basis H which intersects every perfect set of real numbers. Let p be in H and let H/p be the elements of H divided by p. Then 1 is the only rational number in H/p and H/p is a Hamel basis for the reals.

Lemma 1. H/p *intersects every perfect subset of the reals.*

Proof. Suppose P is a perfect set. Then pP is a perfect set and $pP \cap H \neq \emptyset$. Let y be in H and a be in P such that $pa = y$. Then $a = y/p$ is in H/p. So a is in $P \cap H/p$.

Let $G = \{y - [y]: y \text{ is in } H/p \text{ and } y \neq 1\} \cup \{1\}$ where $[y]$ is the greatest integer less than or equal to y. Then G is a subset of I. Since $H/p \cap I$ is a subset of G and intersects every perfect subset of I, G does also. Also G is a Hamel basis for the reals.

Lemma 2. G *contains no perfect subset.*

Proof. Suppose P is a perfect subset of G. Then choose a rational number q such that q lies between two points of P. So $q \neq 1$. Let $M = P \cap [0,q]$ and let $N = P \cap [q,1]$. Now M and N are perfect sets. Assume $q \leq \frac{1}{2}$. Then $M + (1-q)$ is a perfect subset of I. So there exists an x in G such that x is in $M + (1-q)$. Let $x = y + (1-q)$ where y is in M. Since $x - y + (q-1) = 0$, it follows that $1,x,y$ are linearly dependent which is a contradiction. Similarly we have a contradiction, if $q > \frac{1}{2}$.

Lemma 3. *Every Cantor set contains* c *disjoint sub-Cantor sets.*

Proof. Let K be a Cantor set. Then for each x in K, $\{x\} \times K$ is a Cantor set. If $x \neq y$, then $(\{x\} \times K) \cap (\{y\} \times K) = \emptyset$. Thus K^2 contains c disjoint sub-Cantor sets. Since K^2 is a Cantor set and any two Cantor sets are homeomorphic, K contains c disjoint sub-Cantor sets.

Since a Cantor set contains c disjoint sub-Cantor sets, if a set meets every perfect subset in an interval it has cardinality c there. Thus G has the desired properties.

The First Example. Let $\Gamma_1 = \{(x,0): x \text{ is not in } G\}$. Let $\mathcal{K}$ be the collection of closed subsets of $I \times I$ such that $\Pi_x(K)$ has cardinality c for each K in $\mathcal{K}$ where Π_x is the x-projection. Using transfinite induction, select a subset Γ_2 of $I \times I$ such that

(1) Γ_2 intersects each element of $\mathcal{K}$ and

(2) if x and y are in Γ_2, then $\Pi_x(x)$, $\Pi_x(y)$ are in G and $\Pi_x(x) \neq \Pi_x(y)$.

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{(t,1): t \text{ is in } I - \Pi_x(\Gamma_1 \cup \Gamma_2)\}$. Then $\Pi_x(\Gamma) = I$ and Γ is the graph of a function $f: I \to I$.

Suppose f is not a connectivity function. Then Γ is the sum of two mutually separated sets A and B and there exists two mutually exclusive open sets U and V in $I \times I$ such that $A \subset U$ and $B \subset V$. Let $K = I \times I - U \cup V$. Then K contains a continuum L that separates $I \times I$. Choose points P and Q in L such that $\Pi_x(P) \neq \Pi_x(Q)$. Then there exists $(z,0)$ in Γ_1 and $\varepsilon > 0$ such that

(1) the disc $\{(x_1,x_2) \in I^2: |x_1 - z| \leq 2\varepsilon$ and $x_2 \leq 2\varepsilon\} \cap K = \emptyset$ and

(2) the interval $\{x_1 \in I: |x_1 - z| \leq 2\varepsilon\}$ is between $\Pi_x(P)$ and $\Pi_x(Q)$.

Let $S = \{(x_1,x_2) \in I^2: z + \varepsilon \leq x_1 \leq z + 2\varepsilon \text{ and } 0 \leq x_2 \leq 1\}$. Then $K \cap S \subset I^2$ is closed and $(K \cap S) \cap \Gamma = \emptyset$. So $\Pi_x(K \cap S)$ has cardinality less than c. Thus there exists a u such that $z + \varepsilon < u < z + 2\varepsilon$ and u is not in $\Pi_x(K \cap S)$. Hence

$\{(u,t): 0 \leq t \leq 1\}$ separates P from Q in I^2 and does not intersect L. This is a contradiction. Hence $\Pi_x(P) = \Pi_x(Q)$ and L is a proper subset of a vertical interval of I. So L does not separate I^2. This is a contradiction. Therefore Γ is connected and f is a connectivity function.

We now show that f does not have the CIVP. Choose f(x) and f(y) in I such that $f(x) \neq f(y)$. Assume $x < y$ and $f(x) < f(y)$. Let C be a Cantor set in the open interval $(f(x),f(y))$. Now $G \cap (x,y)$ contains no Cantor set but is of cardinality c. So there exists no Cantor set in (x,y) which maps into C.

Definition 3. A subset $S \subset I$ is Cantor dense if and only if for any $0 < a < b < 1$, $[a,b] \cap S$ contains a Cantor set.

Lemma 4. *I is the closure of the union of Cantor dense subsets* S_t *for each* t *in* I *and* $S_r \cap S_s = \emptyset$ *if* $r \neq s$.

Proof. Let $K_{1,1}$ be a Cantor set in $[0,\frac{1}{2}]$ and let $K_{1,2}$ be a Cantor set in $[\frac{1}{2},1]$ such that $K_{1,1} \cap K_{1,2} = \emptyset$.

Let $K_{2,1}$ be a Cantor set in $[0,\frac{1}{4}]$, let $K_{2,2}$ be a Cantor set in $[\frac{1}{4},\frac{1}{2}]$, let $K_{2,3}$ be a Cantor set in $[\frac{1}{2},\frac{3}{4}]$, and let $K_{2,4}$ be a Cantor set in $[\frac{3}{4},1]$ such that the collection $K_{l,m}$ $l = 1,2$ or $m = 1,2,3,4$ are pairwise disjoint.

Continuing this process, let $K_{n,1}$ be a Cantor set in $[0,1/2^n]$, $K_{n,2}$ be a Cantor set in $[1/2^n,2/2^n]$, $\cdots$, and let $K_{n,2^n}$ be a Cantor set in $[(2^n-1)/2^n,1]$ such that $K_{l,m} \cap K_{p,q} = \emptyset$, if $l \neq p$ or $m \neq q$.

Now decompose each $K_{l,m}$ into c disjoint Cantor sets

sets $K^t_{1,m}$ for each t in I. Let $S_t = \cup_{1,m} K^t_{1,m}$. Then S_t is Cantor dense in I. Since S_t is dense in I, the closure of $\cup_t S_t$ is equal to I.

The Second Example. Let g: I → {irrational numbers in I} be 1 - 1 and onto. Define f(x) = g(y) where x is in S_y. If x is not in S_y for each y in I, let f(x) = 0.

Let a and b be in I such that a < b. Suppose f(a) < f(b). Let K be a Cantor set in (f(a),f(b)) and let z be an irrational number in K. Let $w = g^{-1}(z)$. Consider S_w. If x is in S_w, then f(x) = g(w) = z. Thus S_w maps into K. By Cantor density there is a Cantor set $C \subset S_w \cap [a,b]$. Thus $f(C) \subset K$ and f has the CIVP.

Clearly f is not a Darboux function.

Theorem 1. *If* f: I → I *is continuous, then* f *has the* CIVP.

Proof. Assume x and y are in I and x < y. Suppose f(x) < f(y). Let C be a Cantor set in (f(x),f(y)). Let $K = f^{-1}(C) \cap (x,y)$. Since f(x) and f(y) are not in C, x and y are not in $f^{-1}(C)$. Hence $K = f^{-1}(C) \cap [x,y]$ and K is closed. Since K is closed and has cardinality c, K contains a Cantor set P. Since $f(P) \subset C$ we are done.

Theorem 2. *If* f: I → I *is a closed function which has the* CIVP, *then* f *is a continuous function.*

Proof. Choose a and b in I such that a < b. Assume f(a) < f(b). Suppose f is not a Darboux function. Then there exists a y in the open interval (f(a),f(b)) such that if x is in the interval (a,b), then f(x) ≠ y.

Choose a positive integer N such that $[y - \frac{1}{N}, y + \frac{1}{N}]$ is a subset of $(f(a), f(b))$. Now for each Cantor set $C \subset (f(a), y - \frac{1}{N})$, there exists a Cantor set $K \subset (a,b)$ such that $f(K) \subset C$. Choose p in K. Then f(p) is in $(f(a), y - \frac{1}{N})$. Likewise we have a q in (a,b) such that f(q) is in $(y + \frac{1}{N}, f(b))$. Assume $p < q$.

We can construct a collection C_n of Cantor sets such that:

(1) $C_n \subset [y - \frac{1}{N+n}, y + \frac{1}{N+n}] \subset (f(p), f(q))$ where $n = 1, 2, 3, \cdots$,

(2) y is in each C_n,

(3) x_n is in [p,q], $f(x_n)$ is in C_n, and $x_n \neq x_m$ if $n \neq m$, and

(4) $f(x_i)$ is not in C_n if $i = 1, 2, \cdots, n-1$.

Now $\cap C_n = \{y\}$ and $f(x_n)$ converges to y. There exists a subsequence $\{a_n\}$ of $\{x_n\}$ such that a_n converges to some x in $[p,q] \subset (a,b)$. Now $\{a_n\} \cup \{x\}$ is a closed set. Hence $f(\{a_n\} \cup \{x\})$ is closed. Since $f(a_n)$ converges to y and y is not in the set $f(\{a_n\} \cup \{x\})$ we have a contradiction. Therefore f is a Darboux function.

Since a closed Darboux function is continuous, f is a continuous function.

We make the following remarks.

(1) As stated by the referee the function in the first example is almost continuous [7]. A function f is said to be almost continuous if each open set containing f also contains a continuous function with the same domain. Stallings [9] showed that if the function f is almost

continuous and has connected domain, then f is a connectivity function. Examples of connectivity functions which are not almost continuous are given in [1], [3], [6], and [8]. Clearly every continuous function is almost continuous. Thus with previous results and this remark the relations between continuous, almost continuous, connectivity, Darboux, and functions having the CIVP are known.

(2) It follows from lemma 3 that there exists Cantor sets which contain no rational numbers.

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