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1. Introduction

In 1955 A. D. Wallace [9] introduced the study of C-sets and investigated C-sets in semigroups. In this paper we investigate C-sets in Hausdorff continua (compact, connected Hausdorff spaces) and note some properties of C-sets pertaining to the study of mappings onto continua. If M is a continuum, a proper subset H of M is a C-*set* in M provided H is a subset of any subcontinuum of M which contains both a point in H and a point not in H. In Lemma 1 of [9, p. 639] Wallace observed that C-sets are connected and have no interior. Although C-sets do not have to be closed, it is not difficult to show that if K is a C-set which is not closed then $\overline{K}$ is an indecomposable continuum. Moreover, if K is a C-set which is not closed the K is the union of some of the composants of $\overline{K}$. To see this suppose H is a subcontinuum of $\overline{K}$ containing a point in K and a point not in K. Then H contains K and thus H contains $\overline{K}$, so each proper subcontinuum of $\overline{K}$ which intersects K is a subset of K. Consequently, each composant of $\overline{K}$ which intersects K is a subset of K.

A continuum M is a *triod* provided it contains a subcontinuum C such that M-C has at least three components. A continuum is said to be *atriodic* provided it contains no triod. The statement that the continuum M is *unicoherent* means if A and B are continua whose union is M then $A \cap B$ is connected. In his doctoral dissertation at the University of Houston, Collins [2] introduced the class of IUC continua and proved [2, Theorem 6, p. 12] that atriodic continua have property IUC hereditarily. A continuum has *property* IUC provided every proper subcontinuum with interior is unicoherent. Collins' result that atriodic continua have property IUC has been obtained independently by Mackowiak and Tymchatyn [7]. In this paper we generalize these results (Theorem 3).

The so-called "boundary bumping theorem" is used often in the proofs in this paper. For a proof of it in Hausdorff continua see [5, Theorem 2, p. 172].

2. A Characterization of C-Sets

The following theorem, although not stated in this manner, is essentially what Cook [1, Theorem 4, p. 243] and Read [8] (see also [6, 5.7, p. 111]) proved when they showed that a continuum is hereditarily indecomposable if and only if every mapping of a continuum onto it is confluent. The proof presented here differs only slightly and is included only for the sake of completeness.

Theorem 1. *Suppose* M *is a Hausdorff continuum and* H *is a proper subcontinuum of* M. *Then* H *is a* C-*set in* M *if and only if for each mapping* f *of a continuum onto* M *every component of* $f^{-1}(H)$ *is thrown by* f *onto* H.

Proof. Suppose M is a continuum, H is a subcontinuum of M which is a C-set and f is a mapping of a continuum X onto M. Let K be a component of $f^{-1}(H)$ and G be a monotonic collection of subcontinua of X such that the common part of all the members of G is K and each member of G contains a point not in K. Then, if J is a continuum in G, f[J] contains a point of H and a point not in H, so H is a subset of f[J]. Since X is a Hausdorff continuum, G is monotonic and K is the common part of all the members of G, $f[K] = \bigcap_{J \in G} f[J]$. Thus, f[K] = H.

On the other hand suppose H is not a C-set and C is a subcontinuum of M not containing H which contains a point of H and a point Q not in H. Let X be the continuum obtained by identifying (Q,0) and (Q,1) in $(M \times \{0\}) \cup (C \times \{1\})$ and f be the natural projection of X onto M. Then $f^{-1}(H)$ has two components one of which is not thrown onto H by f. This completes the proof of Theorem 1.

Remark. It is easy to show that a continuum is hereditarily indecomposable if and only if every proper subcontinuum of it is a C-set in it.

3. Atriodic Continua and C-Sets

In this section we often use the following property of atriodic continua: If M is a decomposable, atriodic continuum then M is the union of two continua A and B such that $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$. For a proof of this see Collins [2] or [3]. It should be noted that in Collins' work he assumes that continua are metric but his arguments do not require changes for Hausdorff continua.

Lemma. If A *and* B *are two continua which intersect such that* (1) $A \cup B$ *is atriodic,* (2) $A = \overline{A - (A \cap B)}$ *and* $B = \overline{B - (A \cap B)}$, *and* (3) $A \cap B$ *is the union of the two continua* C_1 *and* C_2, *then* A *is irreducible from* C_1 *to* C_2.

Proof. Suppose P is a proper subcontinuum of A which intersects both C_1 and C_2 and let y be a point of A not in $P \cup C_1 \cup C_2$. That there is such a point y follows by the assumption that $A = \overline{A - (A \cap B)}$ for if P contains $A - (C_1 \cup C_2)$ then P contains A. There exist mutually exclusive open sets U_1 and U_2 containing C_1 and C_2 respectively such that $\overline{U}_1$ and $\overline{U}_2$ are mutually exclusive and neither contains y. Let B_1 and B_2 be the components of $B \cap U_1$ and $B \cap U_2$ containing C_1 and C_2 respectively. Then $A \cup (P \cup \overline{B}_1) \cup (P \cup \overline{B}_2)$ is a triod.

The following theorem was proved independently by Maćkowiak and Tymchatyn [7, 13(2), p. 40]. In that paper they call C-sets which are continua terminal continua. This theorem is generalized in the next section of this paper.

Theorem 2. *If* M *is an atriodic continuum then each proper subcontinuum of* M *which is not unicoherent is a* C-*set.*

Proof. Suppose H is a proper subcontinuum of M such that H is not unicoherent and H is not a C-set. Then H is the union of two continua A and B such that $A \cap B$ is not connected and $A = \overline{A - (A \cap B)}$ and $B = \overline{B - (A \cap B)}$. Suppose K is a subcontinuum of M containing a point of $A \cup B$ and a point not in A ∪ B. Since A ∩ B is not connected and M is atriodic, A ∩ B is the union of two continua C_1 and C_2. Suppose K contains a point of B. We now show that K contains A.

Suppose x is a point of A - (A ∩ B) which is not in K. There is an open set U containing x which does not contain a point of B ∪ K. By the Lemma, A is irreducible from C_1 to C_2 so A - (A ∩ U) contains no continuum intersecting both C_1 and C_2. Therefore, [5, Theorem 1, p. 168], A - (A ∩ U) is the union of two mutually exclusive closed point sets H_1 and H_2 containing C_1 and C_2 respectively. There exist mutually exclusive open sets U_1 and U_2 containing H_1 and H_2 respectively such that $\overline{U}_1$ and $\overline{U}_2$ are mutually exclusive and neither contains x. Let A_1 and A_2 denote the components of A ∩ U_1 and A ∩ U_2 containing C_1 and C_2 respectively. Then (B ∪ K) ∪ (B ∪ $\overline{A}_1$) ∪ (B ∪ $\overline{A}_2$) is a triod.

Now, since K contains A, by repeating the argument of the previous paragraph exchanging the roles of A and B, we obtain the K contains B. This will complete the proof.

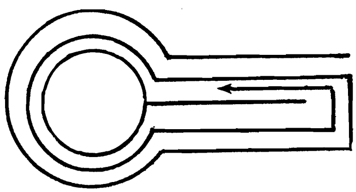
4. HIUC Continua and C-Sets

A continuum having property IUC hereditarily is said to have *property* HIUC.

Theorem 3. *If* M *is a continuum with property* HIUC *then each proper subcontinuum of* M *which is not unicoherent is a* C-*set.*

Proof. Suppose H is a non-unicoherent proper subcontinuum of M, and K is a subcontinuum of M containing a point of H and a point not in H. Suppose x is a point of H which is not in K. Then there is an open set U containing x which contains no point of K. Then H ∪ K does not have property IUC since H is a non-unicoherent proper subcontinuum of H ∪ K which has interior in H ∪ K.

Remark. It is easy to see from the example below that the hypothesis in Theorem 3 that M have property HIUC may not be weakened to M has property IUC for the circle is not a C-set in M.



5. Confluence and Weak Confluence

We conclude this paper with some consequences of Theorems 1, 2 and 3. First, we introduce some terminology which the author has found useful in discussing confluence and related properties.

Definitions. Suppose M is a continuum, H is a subcontinuum of M and f is a mapping of a continuum onto M. The statement that f *is confluent with respect to* H (respectively, *weakly confluent with respect to* H) means each (resp., some) component of $f^{-1}(H)$ is thrown by f onto H.

Thus, if f is a mapping of a continuum onto M then f is *confluent* (resp., *weakly confluent*) provided f is confluent (resp., weakly confluent) with respect to each non-degenerate proper subcontinuum of M. Further, f is said to be *pseudo-confluent* provided f is weakly confluent with respect to each irreducible subcontinuum of M.

Theorem 1 may now be restated: A proper subcontinuum H of a continuum M is a C-set in M if and only if every mapping of a continuum onto M is confluent with respect to H.

The following theorems are immediate from Theorems 1 and 3.

Theorem 4. *Suppose* f *is a mapping of a continuum onto the continuum* M *and* M *has property* HIUC. *If* H *is a non-unicoherent proper subcontinuum of* M *then* f *is confluent with respect to* H.

Theorem 5. *If* f *is a mapping of a continuum onto a continuum* M *having property* HIUC *then* f *is confluent* (resp., weakly confluent) *if and only if* f *is confluent* (resp., weakly confluent) *with respect to every unicoherent proper subcontinuum of* M.

Corollary 1. *Suppose* f *is a mapping of a continuum onto an atriodic continuum. Then* f *is pseudo-confluent if and only if* f *is weakly confluent.*

Finally, we observe that Corollary 1 provides another proof of a theorem of Grispolakis and Tymchatyn [4, Theorem 5.3].

Corollary 2. *Suppose* M *is an atriodic continuum. Then* M *is in Class* W *if and only if* M *is in Class* P.

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