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PRIMALITY OF CERTAIN KNOTS

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This paper proves that the fifteen 4-bridged examples in J. H. Conway's table of 11-crossing knots [2] are each actually prime. Note that we avoid reliance upon assumptions that the prime knot tables are complete or that the minimal crossing number is additive. Cf. [8] and [3].

Reproduced herewith are diagrams of the 552 known 11-crossing primes, of which we here consider knots 12, 84, 220, 225, 240, 357, and 426 through 434. Proof of their primality by another method appears in [4].

Proposition. A knot is prime if (1) *its bridge number* $b(k) \leq 4$ *and* (2) *with respect to the homology of its* 3-*fold dihedral covering spaces* (a) *no* $H_1M_3(k)$ *has odd order and* (b) *the orders of the* $H_1M_3(k)$*'s have no common factor* >3.

Remarks. The first condition may be verified by finding four generators of an entire knot diagram. Compare [1] and [6, p. 606] but beware the concealed conjecture in the latter that bridge number *equals* the minimal number of Wirtinger generators. The second condition must be verified by calculation of $H_1M_3(k)$ for all possible representations of the knot group on S_3 [5]. In the case of each of our 15 examples we get two homology groups, Z_6 and $Z+Z_2$. Note that the existence of a single noncyclic $H_1M_3(k)$ implies that $b(k) > 3$ [1].

Proof. It follows from condition (1) and Schubert's Satz 7 [7]--$b(k_1 \# k_2) = b(k_1)+b(k_2)-1$--that k is prime unless it has a 2-bridged factor. Assume, arguendo, that $k = (\alpha,\beta) \# k'$, where (α,β) is Schubert's normal form notation for 2-bridged knots. Recall that α is odd and >1. Either 3 divides α or 3 does not divide α. We shall derive a contradiction from each of these two possibilities.

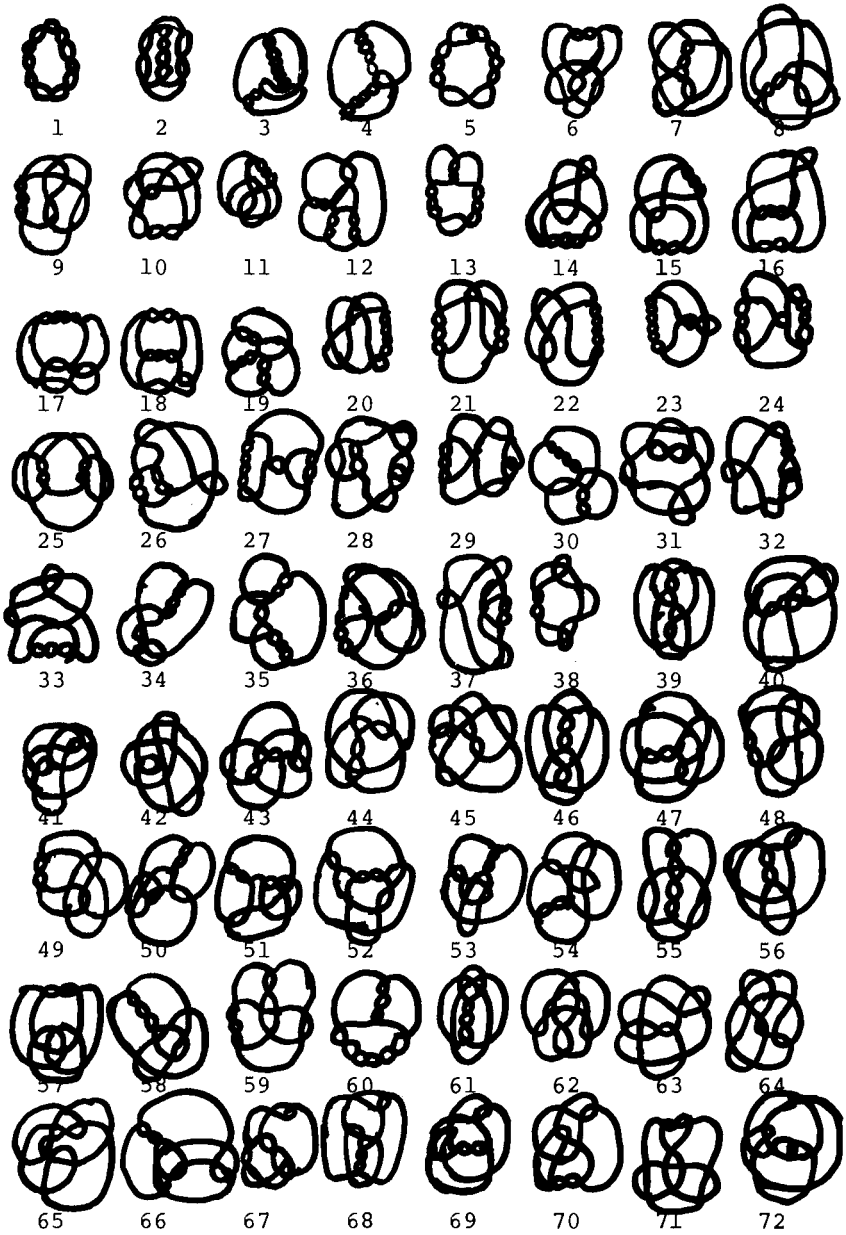
Case A. $3|\alpha$. Then k has a 3-fold irregular covering obtained from the S_3 cover of (α,β) and the constant map on the k' factor--i.e., that which sends all meridians to a single transposition. The homology of such a cover is the same as that of the double branched cover of the knot k'. But the order of the latter, $|H_1M_2(k')| = \Delta(-1)(k')$, is well known to be odd, which violates condition (2)(a).

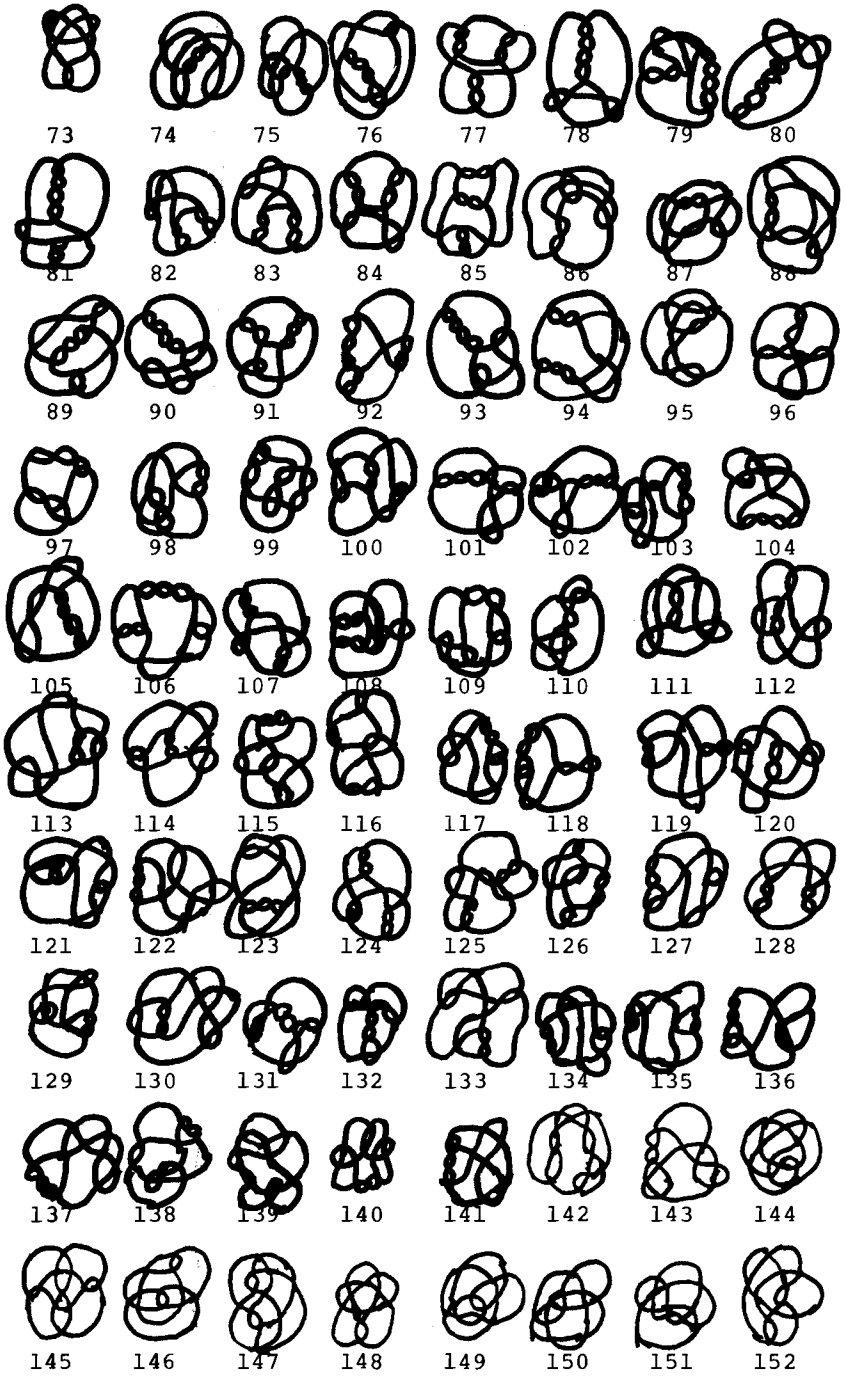
Case B. $3\nmid\alpha$. Here every 3-fold irregular cover of k must be the constant map on the (α,β) factor. By a suitable adjoining of relations derived from the k' factor each such $H_1M_3(k)$ can be shown to admit a surjection on the homology of the 2-fold cyclic cover of (α,β), $H_1M_2(\alpha,\beta) = Z_\alpha$. But this contradicts condition (2)(b). (Indeed, in the case of our 15 examples we get $Z_6 \twoheadrightarrow Z_\alpha$ and $Z+Z_2 \twoheadrightarrow Z_\alpha$ which together imply that $\alpha = 3$.)

Thus k can have no 2-bridged factor and is therefore prime.

PRIME KNOTS WITH ELEVEN CROSSINGS

Drawn by Kathryn Perko

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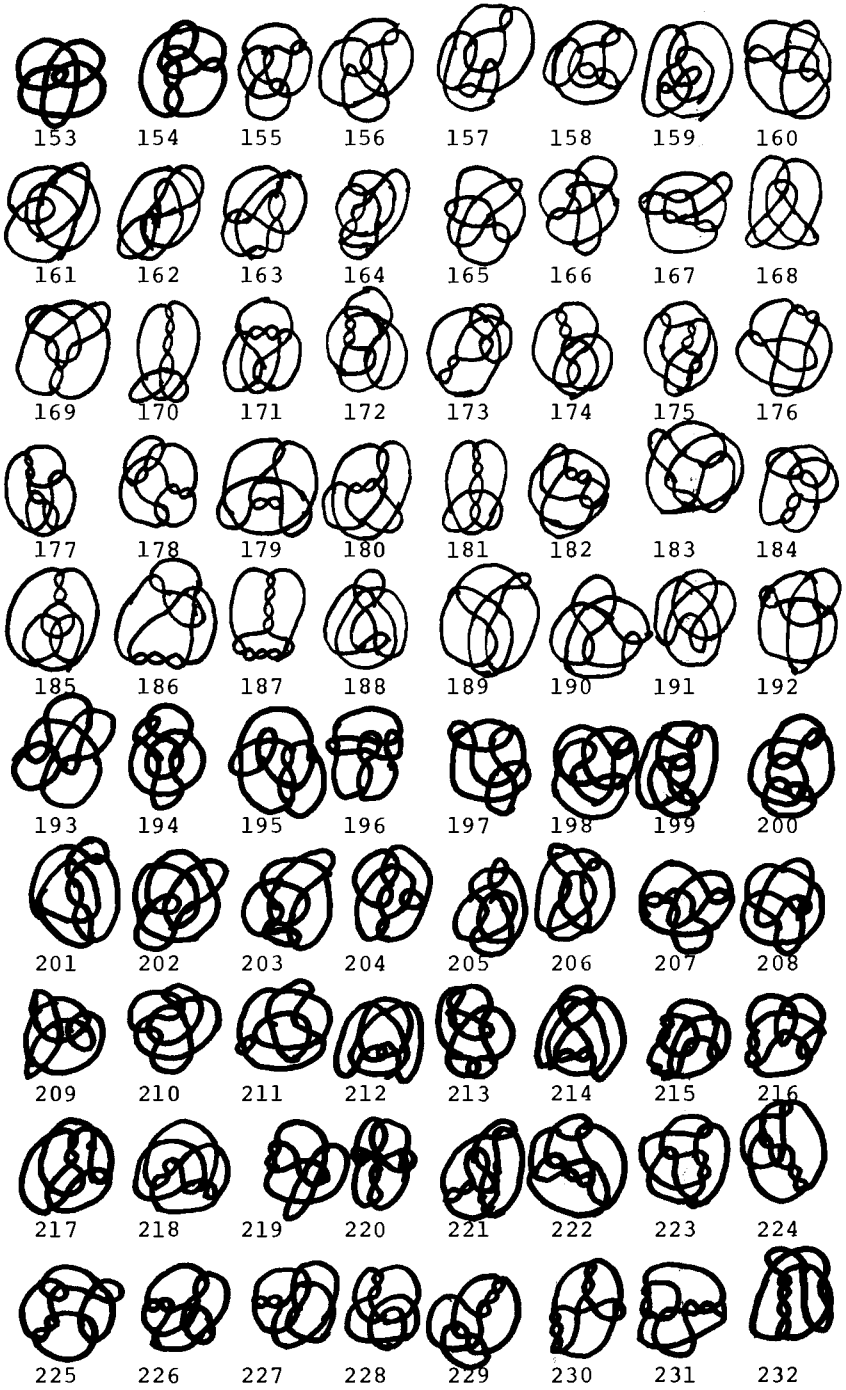
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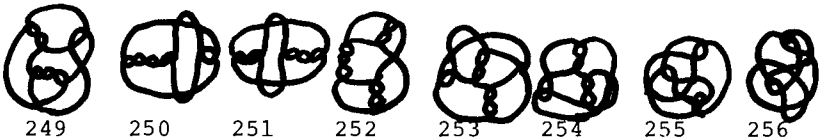
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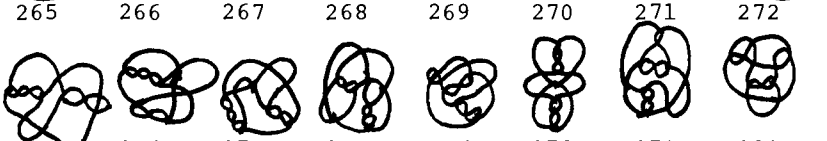
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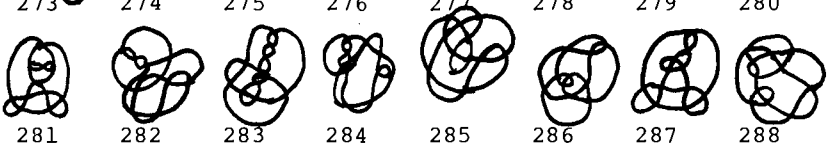
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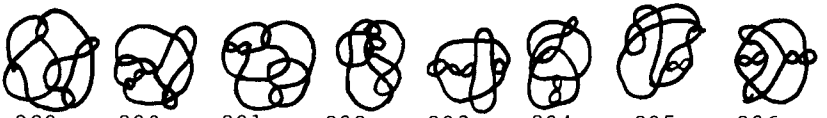
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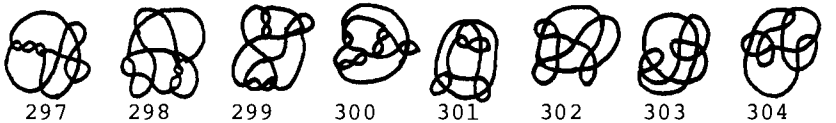
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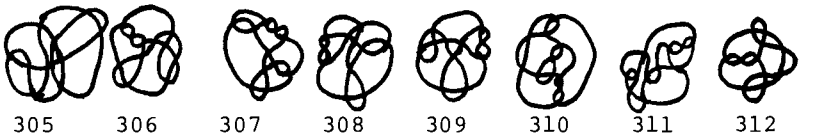
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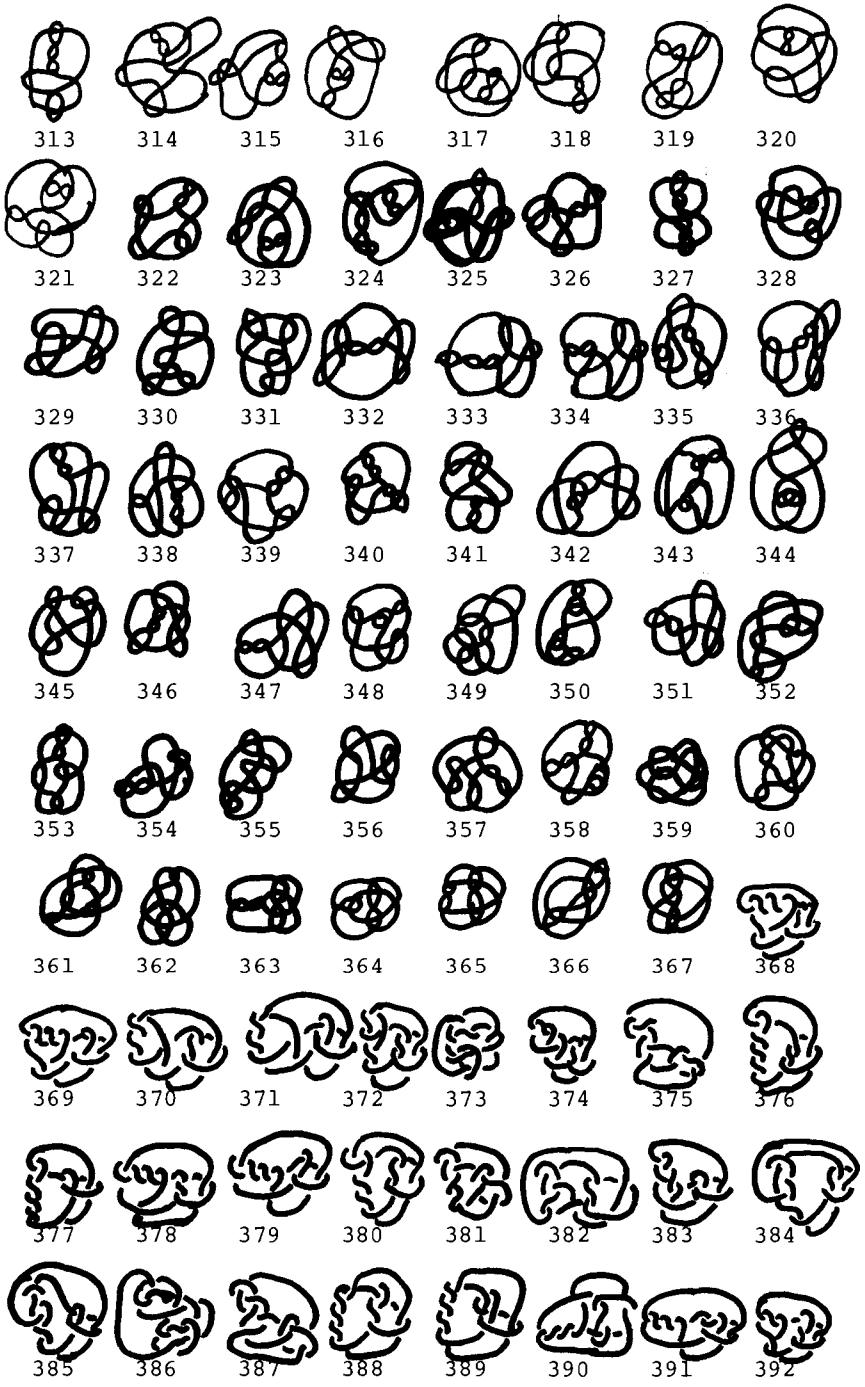
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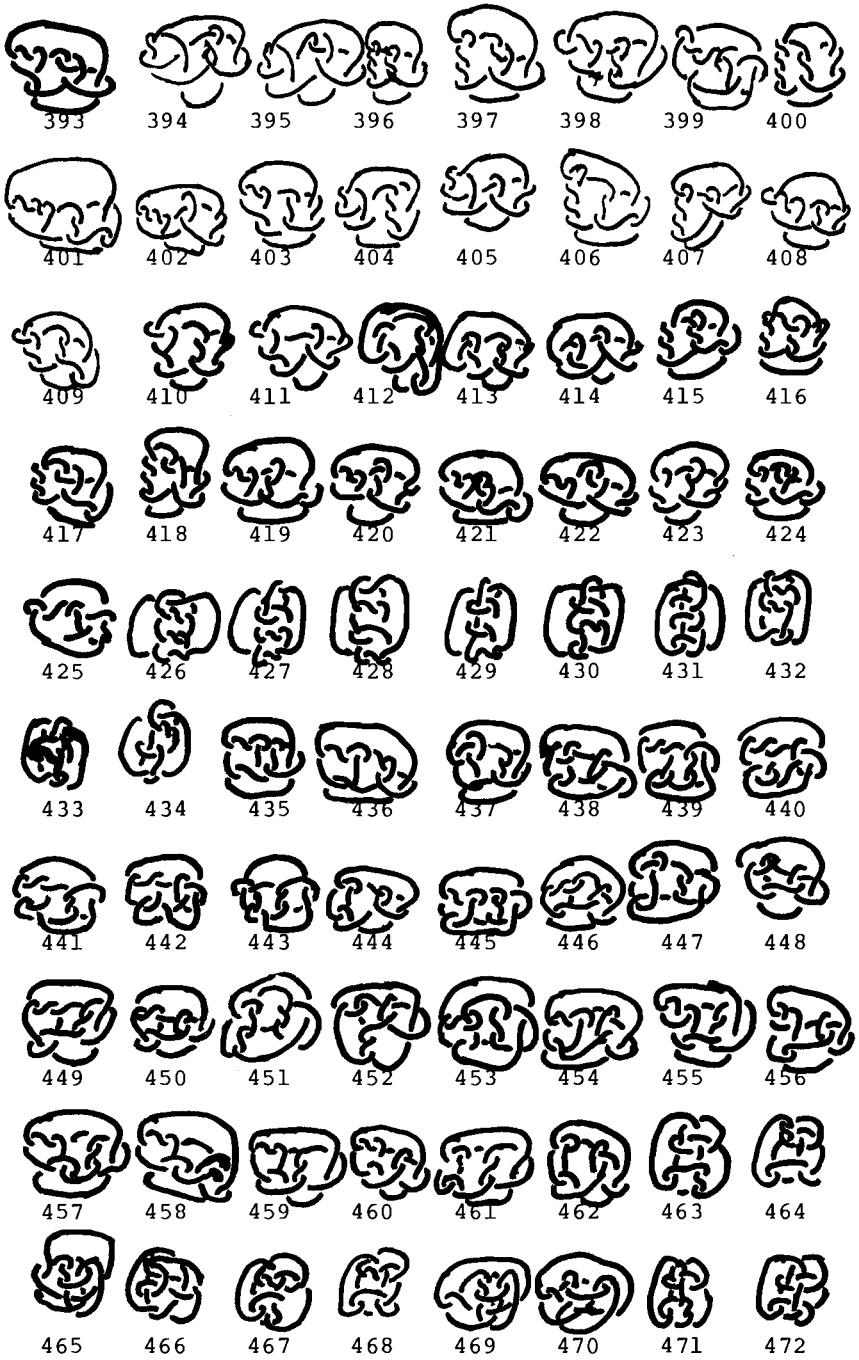
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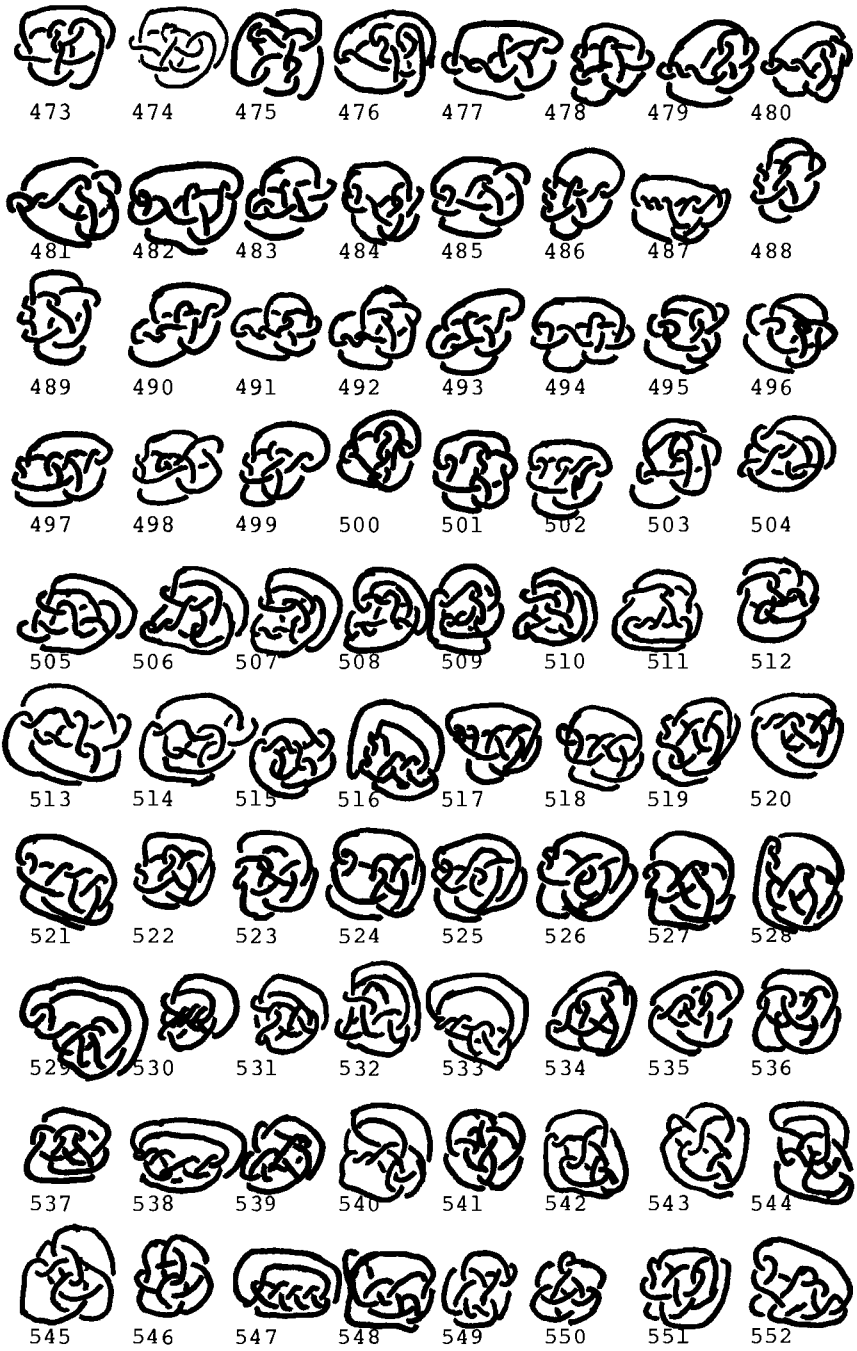
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[1]Note that Conway's 11-crossing table omits knot types 549 through 552. Several thousand 12-crossing knots have recently been classified by Morwen B. Thistlethwaite of the Polytechnic of the South Bank.

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