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by

Choon Jai Rhee

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Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
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ON A CONTRACTIBLE HYPERSPACE CONDITION

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Let X be a metric continuum with metric d. Denote by 2^X and C(X) the hyperspaces of non-void closed subsets and subcontinua of X respectively and endow each with the Hausdorff metric H. Let μ: C(X) $\to$ [0,1] be a Whitney map [5] with $\mu(X) = 1$.

We define the total fiber at a point $x \in X$ to be the set $F(x) = \{A \in C(X) | x \in A\}$, and an element $A \in F(x)$ is said to be admissible at x if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that each point y in the δ-neighborhood of x has an element $B \in F(y)$ such that $H(A,B) < \varepsilon$. Let $\mathcal{a}(x) = \{A \in F(x) | A$ is admissible at x$\}$, and let $\mathcal{a}_t(x) = \mathcal{a}(x) \cap \mu^{-1}(t)$, for $0 \leq t \leq 1$. We say that the space X is admissible if $\mathcal{a}_t(x) \neq \emptyset$ for each $(x,t) \in X \times [0,1]$.

In [1], Kelley has a sufficient condition, namely property 3.2, for the contractibility of the hyperspace C(X). In [4], Wardle has a pointwise version of this property which was subsequently named as property K at a point in [2]. In [4], it was shown that for a metric continuum having property K at each of its points is equivalent to property 3.2. He also showed that the set-valued map $x \to F(x) \subset C(X)$ is upper-semicontinuous for any metric continuum X and is continuous if and only if X has property K at each of its points. In [3] we showed that the admissibility condition on X is a necessary condition in order that C(X) is contractible. In the present paper we give some additional conditions on the collection $\mathcal{a}(x)$ and show that the map $x \to \mathcal{a}(x)$ is a lower-semicontinuous set-valued map and the contractibility of C(X) is equivalent to a selection of a continuous fiber map.

Finally, we exhibit two examples to demonstrate the notion, and state some open questions.

1. Preliminaries

We collect in this section some definitions and known facts. Let (X,d) be a metric continuum.

Definition 1.1 [1]. X has property K if, for each $\varepsilon > 0$ there is $\delta > 0$ such that if $a,b \in X$, $d(a,b) < \delta$ and $a \in A \in C(X)$ then there exists B, $b \in B \in C(X)$ with $H(A,B) < \varepsilon$.

Definition 1.2 [4]. X is said to have property K at a point x provided that for each $\varepsilon > 0$ there is $\delta > 0$ such that if $y \in X$, $d(x,y) < \delta$, and $x \in A \in C(X)$ then there exists B, $y \in B \in C(X)$ with $H(A,B) < \varepsilon$.

In [4], it was already mentioned that X has property K if and only if X has property K at each of its points.

Lemma 1.3 [1]. *The following statements are equivalent.*

(1) *A contraction of* X *in* 2^X *exists*

(2) 2^X *is contractible*

(3) C(X) *is contractible.*

Proposition 1.4 [3]. *For each* $x \in X$ *its admissible fiber* $\mathcal{a}(x)$ *is closed in* C(X), $\{x\} \in \mathcal{a}(x)$ *and* $X \in \mathcal{a}(x)$.

2. Fiber Maps

Let X be a metric continuum. A set-valued map $\alpha: X \to C(X)$ is said to be a fiber map if, for each $x \in X$, $\alpha(x)$ is a closed subset of the admissible fiber $\mathcal{a}(x)$ at x such that

(I) $\{x\}$, X are members of $\alpha(x)$

(II) for each pair A_0, A_1 in $\alpha(x)$, $A_0 \subset A_1$, there is an ordered arc [2, p. 57] in $\alpha(x)$ from A_0 to A_1.

(III) for each $A \in \alpha(x)$, and $\varepsilon > 0$, there is a neighborhood W of x such that each point y in W has an element $B \in \alpha(y)$ such that $H(A,B) < \varepsilon$.

It is clear that the condition (I) and (II) imply the admissibility of the space X. Let us say that the space X has property c if there is a fiber map α on X into C(X).

Proposition 2.1. *Every fiber map* $\alpha: X \to C(X)$ *is lower-semicontinuous.*

Proof. Let $x \in X$ and $\varepsilon > 0$. Since $\mathcal{a}(x)$ is compact by proposition 1.4 $\alpha(x)$ is compact so we choose $A_1, \cdots, A_n \in \alpha(x)$ such that if A is any element of $\alpha(x)$, then $H(A, A_{i_0}) < \varepsilon/2$ for some $1 \leq i_0 \leq n$. Applying condition (III) to each A_j, $j = 1, \cdots n$, we find $\delta_j > 0$ such that each point y in the δ_j-neighborhood of x has an element $B_j \in \alpha(y)$ such that $H(A_j, B_j) < \varepsilon/2$. Let $\delta = \min\{\delta_j | j = 1, \cdots, n\}$, and let y be any point of the δ-neighborhood of x, and let $B_{i_0} \in \alpha(y)$ such that $H(A_{i_0}, B_{i_0}) < \varepsilon/2$. Then $H(A, B_{i_0}) \leq H(A, A_{i_0}) + H(A_{i_0}, B_{i_0}) < \varepsilon$. Therefore α is lower-semicontinuous at x.

Proposition 2.2. *Suppose* $\alpha: X \to C(X)$ *is a fiber map. Let* $\bar{\alpha}(x,t) = \alpha(x) \cap \mu^{-1}(t)$, $0 \leq t \leq 1$. *Then* $\bar{\alpha}: X \times I \to C(X)$ *is lower-semicontinuous.*

Proof. Let $(x_0,t_0) \in X \times [0,1]$ and $\varepsilon > 0$. Since $\bar{\alpha}(x,t)$ is compact, in view of the above, it suffices to show that if N_ε is an ε-neighborhood of $A_0 \in \bar{\alpha}(x_0,t_0)$, there is an open set $W \times J \subset X \times I$ containing (x_0,t_0) such that $\bar{\alpha}(x,t) \cap N_\varepsilon \neq \emptyset$ for each $(x,t) \in W \times J$.

Let $\delta = \delta(\frac{\varepsilon}{4}) > 0$ be a number which is provided by Lemma 1.5 in [1] such that if $|\mu(B) - \mu(B')| < \delta$, $B \subset B'$, $B, B' \in C(X)$, then $H(B,B') < \varepsilon/4$. (1) Since μ is uniformly continuous, there is a $\varepsilon_0 > 0$ such that if $H(B,B') < \varepsilon_0$, then $|\mu(B) - \mu(B')| < \delta$. (2)

Let J be the δ-neighborhood of t_0 in [0,1], and let $\tau = \min\{\frac{\varepsilon}{4},\varepsilon_0\}$. Applying τ to A_0 in the condition (III), one finds a neighborhood W of x_0 such that each point y in W has an element $B \in \alpha(y)$ such that $H(A_0,B) < \tau$. (3)

Now suppose $(x,t) \in W \times J$ and $B' \in \alpha(x)$ which satisfies (3) above. Then $H(A_0,B') < \tau < \varepsilon_0$ implies $|\mu(A_0) - \mu(B')| < \delta$ by (2) above. So let $s_0 = \mu(B')$. Then $s_0 \in J$.

Now suppose $s_0 \leq t_0$. Applying the condition (II), one finds an element $B \in \alpha(x)$ such that $t_0 = \mu(B)$ and $B' \subset B$. Thus $H(B',B) < \varepsilon/4$ by (1) above.

Since $|t_0 - t| < \delta$, we apply the condition (II) again to get an element $C \in \alpha(x)$ ($C \subset B$ if $t < t_0$, $C \supset B$ if $t > t_0$) such that $H(C,B) < \varepsilon/4$ and $\mu(C) = t$. So $H(A_0,C) \leq H(A_0,B') + H(B',B) + H(B,C) < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon$. Hence $\bar{\alpha}(x,t) \cap N_\varepsilon \neq \emptyset$. If $t_0 \leq s_0$, then the proof is similar.

Theorem 2.3. *Let* X *be a metric continuum with property* c. *Then* C(X) *is contractible if and only if there is a continuous fiber map* α *such that* $\alpha(x) \subset \mathcal{a}(x)$ *for each* $x \in X$.

Proof. Suppose C(X) is contractible. Then by Lemma 1.3, there is a single-valued contraction $h: X \times [0,1] \to C(X)$ such that $h(x,0) = \{x\}$, $h(x,1) = X$, and $h(x,t) \subset h(x,t')$ if $t \leq t'$, for each $x \in X$ and $t \in [0,1]$. Since h(x,t) is an admissible element at x [3], $h(x,t) \in \mathcal{a}(x)$ for each $x \in X$. Let $\alpha(x) = \{h(x,t) | 0 \leq t \leq 1\}$. Then it is clear that α satisfies the conditions (I), (II), (III).

Conversely, suppose $\alpha: X \to C(X)$ is a continuous fiber map such that $\alpha(x) \subset \mathcal{a}(x)$. Since $\{x\}, X \in \alpha(x)$, $\alpha(x) \cap \mu^{-1}(t) \neq \emptyset$ by (I) and (II) respectively, we have a set-valued map $\overline{\alpha}: X \times [0,1] \to C(X)$, defined by $\overline{\alpha}(x,t) = \alpha(x) \cap \mu^{-1}(t)$.

Suppose that $\{(x_n,t_n)\}$ is a sequence which converges to (x_0,t_0), and $A_n \in \overline{\alpha}(x_n,t_n)$ such that the sequence converges to A_0. Then by the continuity of α, $A_0 \in \alpha(x)$ and by the continuity of μ, $A_0 \in \mu^{-1}(t_0)$. Therefore lim sup $\overline{\alpha}(x_n,t_0) \subset \alpha(x_0,t_0)$. Hence $\overline{\alpha}$ is upper semicontinuous at (x_0,t_0). This together with Proposition 2.2, we have $\overline{\alpha}$ is continuous at (x_0,t_0).

Now let $C^2(X)$ be the hyperspace of subcontinua of C(X), and let $\sigma: C^2(X) \to C(X)$ be the union map [1], i.e. $\sigma(\mathcal{a}) = \cup\{A | A \in \mathcal{a}\}$. Then we define $h(x,t) = \sigma \circ \overline{\alpha}(x,t)$. $h: X \times [0,1] \to C(X)$ is continuous and it is easy to verify that h is a contraction.

In [3], we defined the $\mathcal{M}$-set of a metric continuum X to be the set $M = \{x \in X | \alpha(x) \neq F(x)\}$, and points of $X \backslash M$ are called K-points of X.

Proposition 2.4 [4]. *Let* X *be a metric continuum. The following are equivalent.*

(1) *The total fiber map* $F: X \to C(X)$ *is continuous at* x

(2) X *has property* K *at* x

(3) x *is a* K-*point of* X

Proof. (1) $\Leftrightarrow$ (2) by Theorem 2.2 in [4].

(2) $\Rightarrow$ (3). It is obvious that property K at x implies that x is a K-point of X.

(3) $\Rightarrow$ (2). Since $F(x) = \alpha(x)$, and $\alpha(x)$ is compact by Proposition 1.4 for given $\varepsilon > 0$, we find elements $A_1,\cdots,A_n$ in $\alpha(x)$ such that each element A in $\alpha(x)$ is less than $\varepsilon/2$ apart from A_i, for some i. Therefore, for each i, $i = 1,2,\cdots,n$, there is $\delta_i > 0$ such that each point y in the δ_i-neighborhood of x has an element $B_i \in F(y)$ such that $H(A_i,B_i) < \varepsilon/2$. So, let $\delta = \min\{\delta_i | i = 1,\cdots,n\}$ and y a point of the δ-neighborhood of x. Then $0 < \delta < \delta_i$, so that $H(A,B_i) \leq H(A,A_i) + H(A_i,B_i) < \varepsilon$ for some i, and $B_i \in F(y)$. Therefore X has property K at x.

3. Examples

In this section we give two examples of spaces having property c each of which possesses a non-void connected $\mathcal{M}$-set and a continuous fiber map α.

Since there is no selection theorems for lower-semi-continuous fiber maps, our technique of selecting continuous fiber map α in our examples depends on specific character of spaces. However, in view of Propositions 2.4, 3.2 and Corollary 3.3 in [3], it is preferable to look at it as an extension α of a continuous fiber α': $M \to C(X)$ over X such that $\alpha(x) \subset a(x)$ for each $x \in X$.

Example 3.1 [1]. Let Y be the graph of $(\text{Sin}\,\frac{1}{t} + 1)$, $0 < t \leq 1$, $\overline{P_0q_0}$ be the segment joining the points $P_0 = (0,0)$ and $q_0 = (0,2)$, and $\overline{\gamma_0P_0}$ be the segment joining the points $\gamma_0 = (0,-1)$ and P_0. Let $X = Y \cup \overline{P_0q_0} \cup \overline{\gamma_0P_0}$. It is known in [1] that the hyperspaces of X are contractible. Let $M = \overline{P_0q_0}\setminus\{q_0\}$, and $x \in M$, $A \in F(x)$. Then $A \in a(x)$ if and only if $A \subset \overline{P_0q_0}$ or $A \supset \overline{P_0q_0}$. Since there are elements in $F(x)\setminus a(x)$, x is a point of the $\mathcal{M}$-set of X. If $x \in X\setminus M$, then x is a K-point. Hence M is the $\mathcal{M}$-set of X. Furthermore, it is easy to verify that the set-valued map a: $X \to C(X)$ is a fiber map, and the restriction α' of a on M is continuous. Let $F(M) = \{A \in C(X) \mid M \subset A\}$. Extend α' to α: $X \to C(X)$ by

$$\alpha(x) = \{\overline{xa} \in C(\overline{\gamma_0q_0}) \mid \overline{xa} \text{ is a segment between } x \text{ and } a\text{, and } a \text{ is between } x \text{ and } q_0\} \cup \{A \in F(M) \mid x \in A\},\ x \in \overline{\gamma_0P_0}\setminus\{P_0\},$$
$$= a(x),\ x \in \overline{Y}.$$

Then $\alpha(x) \subset a(x)$ for each $x \in X$, and α is a fiber map. Since each point x of $Y \cup \{q_0\}$ is a K-point of X, $\alpha(x) = a(x) = F(x)$, is continuous at each point of $Y\setminus\{q_0\}$. Also it can be easily checked that α is continuous at each point of $\overline{\gamma_0P_0}\setminus\{P_0\}$.

Suppose $x_0 \in M$ and $\{x_n\}$ is a sequence in $X\backslash M$ which converges to x_0, $A_n \in \alpha(x_n)$ such that $\{A_n\}$ converges to an element $A_0 \in C(X)$. Then $x_0 \in A_0$, and either $A_0 \in F(M)$ or $A_0 \subset \overline{P_0 q_0}$. Hence $A_0 \in \alpha(x_0) = \{A \in C(\overline{P_0 q_0}) | x_0 \in A\} \cup \{A \in F(M) | x_0 \in A\}$. Therefore α is continuous at x_0. Thus α is a continuous fiber map.

Example 3.2. Let $X = (\bigcup_{n=1} L_n) \cup S$, where S is the unit circle center at the origin, and L_n is an arc defined in the polar coordinates $L_n = \{(\gamma,\theta) | \gamma = (1 + \frac{1}{n}) - \theta/2\pi n, 0 \leq \theta \leq 2\pi\}$, $n = 1,2,3,\cdots$.

Since X is locally connected at each point of $X|S$, we have $\alpha(x) = F(x)$, for $x \in X|S$ by Proposition 3.4 [3]. Hence α is continuous at $x \in X|S$ by Proposition 2.4.

Let $x_0 \in S$, and A is an arc in S such that x_0 is an end point of A. Then either $A \in \alpha(x_0)$ or $\overline{S\backslash A} \in \alpha(x_0)$, but not both belong to $\alpha(x_0)$. Hence the $\mathcal{M}$-set of X is S. It is not difficult to see that the set-valued map $\alpha: X \to C(X)$ is a fiber map which is continuous at each point of $X|S$, and discontinuous at each point of S. Let $P_0 = (1,0)$, and $F(S) = \{A \in C(X) | A \supset S\}$. We denote the counter clockwise orientation on X by ω. For $x \in S$, let $\beta_0(z) = \{\overline{xz} | \overline{xz}$ is the unique arc in S from z to x with respect to ω, $x \in S\backslash\{z\}\} \cup \{S,\{z\}\}$, and $\gamma_0(z) = \{A \in F(S) | z \in A\}$. For $z \in L_n$, let $\beta_n(z) = \{\overline{xz} | \overline{xz}$ is the unique arc in L_n from z to x with respect to ω, $x \in L_n\backslash\{z\}\} \cup \{\{z\}\} \cup \{\overline{P_0 z} \cup A | A \in \beta_0(P_0)\}$, and let $\gamma_n(z) = \{A \in F(S) | z \in A\}$. It is easy to verify that $\beta_j(z)$ and $\gamma_j(z)$ are both closed subsets of $C(X)$, $j = 0,1,2,\cdots$, and the set-valued maps β and

γ defined by

$$\beta(z) = \begin{cases} \beta_0(z), & z \in S \\ \beta_n(z), & z \in L_n, \end{cases} \qquad \gamma(z) = \begin{cases} \gamma_0(z), & z \in S \\ \gamma_n(z), & z \in L_n \end{cases}$$

are both continuous on X.

Let $\alpha(z) = \beta(z) \cup \gamma(z)$, $z \in X$. Then α is continuous. It is easily verified that α is a fiber map.

Question 1. Does the admissibility of a space X imply property c?

Question 2. For each fiber map $\alpha: X \to C(X)$, does there exist a continuous fiber map $\beta: X \to C(X)$ such that $\beta(x) \subset \alpha(x)$, for $x \in X$?

Affirmative answers to both of the questions provide a complete classification of metric spaces having contractible hyperspaces.

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Wayne State University

Detroit, Michigan 48202