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TOTALLY DISCONNECTED SPACES AND INFINITE COHOMOLOGICAL DIMENSION

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1. Introduction

Do all infinite dimensional (separable metric) spaces have infinite cohomological dimension? This question has been of recent interest, especially in the case of compacta, although it is no less interesting for the case of arbitrary spaces. In [W1] John Walsh showed the existence of a wide class of compacta having infinite cohomological dimension (this was somewhat generalized in [Bo]). From Walsh's work in [W2] it can be deduced that any space containing subspaces of arbitrarily high finite dimension must itself have infinite cohomological dimension. In general it seems to be difficult to determine the cohomological dimension of a space that "does not contain finite dimensional subspaces" unless it is constructed according to the principles described in [W1].

The study of cohomological dimension has been motivated very much by results of R. D. Edwards. These are discussed in the aforementioned paper [W2] to which the reader may turn for more enlightenment. To review the matter, recall that it is an open problem whether there exists a cell-like map between compacta that raises dimension. In Section 6 of [W2] we have the beautiful theorem,

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1.1. *Theorem.* (Vietoris; R. D. Edwards) *A compactum has cohomological dimension* $\leq n$ *if and only if it is the image of a cell-like map defined on a compactum having dimension* $\leq n$.

For finite dimensional spaces, cohomological dimension agrees with dimension (see Theorem 3.2 (b) of [W2]). We are left with the possibility of the existence of an infinite dimensional compactum (noncompact space?) with finite cohomological dimension; by 1.1 that compactum would be the image of a dimension raising cell-like map.

Results in Section 3 below will show that there is a connection between compacta and noncompacta in the study of cohomological dimension. In Section 4 three classes of hereditarily strongly infinite dimensional spaces will be introduced. One of these will be demonstrated to have infinite cohomological dimension. For the other two our techniques do not seem to prevail--thus, calculating their cohomological dimension remains an open problem.

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2. Preliminaries

The Hilbert Cube Q is $\prod\{I_k | k = 1,2,\cdots\}$ where $I_k = [-1,1]$. Let $\pi_k\colon Q \to I_k$ be the coordinate projection, $A_k = \pi_k^{-1}(-1)$, $B_k = \pi_k^{-1}(1)$. For $0 \leq t \leq \frac{1}{2}$, let $A_k^t = \pi_k^{-1}([-1,-1+t])$ and $B_k^t = \pi_k^{-1}([1-t,1])$. As usual we treat the n-cube I^n as a subspace of Q, and $S^n = \partial I^{n+1}$. Then sometimes we write A_k for $A_k \cap I^n$ and B_k for $B_k \cap I^n$. The

Eilenberg-MacLane space $K_n = K(Z,n)$ is described in [W2]. We treat K_n as a CW-complex such that $S^n \subset K_n$ and,

$$\pi_k(K_n,*) \simeq \begin{cases} \pi_k(S^n,*) & k \leq n \\ 0 & k \geq n+1. \end{cases}$$

2.1. *Definition.* A map $f: X \to I^{n+1}$ is *stable* if $f|f^{-1}(S^n): f^{-1}(S^n) \to S^n$ does not extend to a map of X to S^n. It is called *cohomologically stable* if it does not extend to a map of X to K_n.

2.2. *Note.* The existence of a stable map is equivalent to dim $X \geq n + 1$, while that of a cohomologically stable map implies c-dim $X \geq n + 1$ (c-dim means cohomological dimension). See [W2] pp. 106-107.

2.3. *Notation.* If Γ is a set of natural numbers, then by Q_Γ we mean the set of all $(x_1,x_2,\cdots,x_i,\cdots) \in Q$ such that $x_i = 0$ if $i \notin \Gamma$. Thus if Γ is finite, then Q_Γ is a copy of I^n for some n.

2.4. *Definition.* A collection $\{(A_k',B_k') | k \in \Gamma\}$ of disjoint pairs of closed subsets of a space X is called an *essential family* for X provided that if S_k is a closed set separating A_k' and B_k' in X for each $k \in \Gamma$, then $\cap\{S_k | k \in \Gamma\} \neq \emptyset$. (This implies dim $X \geq$ card Γ.) A set such as S_k is often called a *separator* of (A_k',B_k').

The folloiwng Proposition is similar to 5.5 of [R-S-W]; the first part does not require compactness.

2.5. *Proposition. Let* $\{(A_k',B_k') | k \in \Gamma\}$ *be an essential family for a space* X *and let* $J \subset \Gamma$. *If for each*

$j \in J$, S_j *is a separator of* (A_j', B_j') *and* $X^* = \cap\{S_j | j \in J\}$, *then* $\{(A_k' \cap X^*, B_k' \cap X^*) | k \in \Gamma - J\}$ *is an essential family for* X^*. *If in addition* X *is compact, then for each* $k \in \Gamma - J$, X^* *contains a continuum meeting* A_k' *and* B_k'.

2.6. *Definition.* A space is called *strongly infinite dimensional* if it has an infinite essential family. It is *hereditarily strongly infinite dimensional* if in addition each subspace is either 0-dimensional or strongly infinite dimensional. Such spaces are constructed in [Ru1, Ru2].

Throughout this paper, spaces are separable and metrizable. Thus they all can be embedded topologically in Q.

3. A Totally Disconnected Space

There exists a totally disconnected, strongly infinite dimensional G_δ-space Y having the property that every strongly infinite dimensional compactum contains a copy of a closed, hereditarily strongly infinite dimensional subspace of Y. Hence if every hereditarily strongly infinite dimensional closed subspace of Y were of infinite cohomological dimension, then every strongly infinite dimensional compactum would have infinite cohomological dimension.

Let $C \subset I_1$ be a Cantor set in the interior of $[-\frac{1}{2}, \frac{1}{2}]$, and let $Y \subset Q$ be a space consisting of at least one point from each continuum in Q that meets both A_1 and B_1 and such that $\pi_1 : Q \to I_1$, when restricted to Y, is a bijection of Y onto C (see 4.5 of [R-S-W]). Thus Y is a totally disconnected space, and it is known that Y can be chosen to be topologically complete; i.e., Y is a G_δ-space [Pol]. Such are the spaces from which Roman Pol constructed his amazing example of an infinite dimensional compactum which is neither countable dimensional nor strongly infinite dimensional. The family $\{(A_i^t \cap Y, B_i^t \cap Y) | i = 2,3,\cdots\}$ for any $0 < t \leq \frac{1}{2}$ is an essential family in Y, and so Y is strongly infinite dimensional. To see why this is so, let S_i, $i \geq 2$, be a separator in Y of $(A_i^t \cap Y, B_i^t \cap Y)$. As in the proof of Theorem 6.2 of [Rul], there are sets $\tilde{S}_i$ closed in Q, separating A_i and B_i in Q, and such that $\tilde{S}_i \cap Y = S_i$. By 2.5, $\cap\{\tilde{S}_i | i = 2,3,\cdots\}$ contains a continuum meeting A_1 and B_1, so $\emptyset \neq Y \cap (\cap\{\tilde{S}_i | i = 2,3\cdots\}) = \cap\{(Y \cap \tilde{S}_i) | i = 2,3,\cdots\} = \cap\{S_i | i = 2,3\cdots\}$. Thus by definition, $\{(A_i^t \cap Y, B_i^t \cap Y) | i = 2,3,\cdots\}$ is an essential family for Y.

For any subset Γ of the set of natural numbers, let $Y_\Gamma = Y \cap Q_\Gamma$ (see 2.3). Then an argument similar to that just given proves the following.

3.1. *Proposition. For* $0 < t \leq \frac{1}{2}$ *and any subset* Γ *of the natural numbers such that* $1 \in \Gamma$*, the collection* $\{(A_i^t \cap Y_\Gamma, B_i^t \cap Y_\Gamma) | i \in \Gamma$ *and* $i \geq 2\}$ *is an essential family for the totally disconnected space* Y_Γ.

3.2. *Proposition. Let* N_0 *be an infinite subset of the set of natural numbers with* $1 \notin N_0$ *and suppose* $0 < t \leq \frac{1}{2}$. *Then there exists a set* $\{Z_k | k \in N_0\}$ *of closed subsets of* Q *satisfying,*

3.2.1. Z_k *continuum-wise separates* A_k^t *and* B_k^t,

3.2.2. $Z_k \cap A_k = \emptyset$ *and* $Z_k \cap B_k = \emptyset$, *and*

3.2.3. $Z = \cap\{Z_k | k \in N_0\}$ *is hereditarily strongly infinite dimensional.*

Proof. See the proof of 3.1 of [Ru2].

3.3. *Theorem. If* K *is a strongly infinite dimensional compactum, then* K *contains a totally disconnected subspace homeomorphic to a hereditarily strongly infinite dimensional closed subspace of* Y.

Proof. Let K have essential family $\{(A_i', B_i') | i \geq 1\}$. Embed K in Q so that $A_1' \subset A_1$, $B_1' \subset B_1$, while $A_i' \subset A_{2(i-1)}$, $B_i' \subset B_{2(i-1)}$, $i \geq 2$. For example, one may use the Embedding Lemma 3.5 of [K] and then perhaps some renaming of coordinates. Let $N_0 = \{2k | k \geq 2\}$ and choose $\{Z_k | k \in N_0\}$ and $t = \frac{1}{2}$ as in 3.2. Let $X = Y \cap Z \cap K$. If X is not 0-dimensional then 3.2.3 implies that X is hereditarily strongly infinite dimensional as required. So we need only show that X is not 0-dimensional.

To this end it will be demonstrated that $\{(A_2^t \cap X, B_2^t \cap X)\}$ is an essential family for X. For let S_2 be a separator of $(A_2^t \cap X, B_2^t \cap X)$ in X. Extend S_2 to a set Z_2 closed in Q so that $X \cap Z_2 = S_2$, while Z_2 is a separator of A_2 and B_2 in Q. By 2.5, $Z \cap K \cap Z_2$ contains a continuum meeting A_1 and B_1, so $\emptyset \neq Y \cap Z \cap K \cap Z_2 = X \cap Z_2 = S_2$. Hence $\{(A_2^t \cap X, B_2^t \cap X)\}$ is an essential family for X, and the proof is complete.

4. Calculating Cohomological Dimension

Let Y be a strongly infinite dimensional, totally disconnected space chosen as in Section 3. Let

$N_0 = \{2k | k \geq 1\}$ and $t = \frac{1}{2}$; choose $\{Z_k | k \in N_0\}$ and Z as in 3.2. With $X = Y \cap Z$, then using 3.1, 3.2 and 2.5 we see that X is a hereditarily strongly infinite dimensional, totally disconnected space. Define X_1 to be the closure of X in Q, and then let $X_2 = \cap\{\mathrm{cl}_Q(Z_k \cap Y) | k \in N_0\}$. Then X_1, X_2 are hereditarily strongly infinite dimensional compacta.

4.1. *Theorem. For any space* X_2 *chosen as in the preceding paragraph*, c-dim $X_2 = \infty$.

In order to prove 4.1, we will need some preliminary lemmas. The reader who is familiar with the proof of Theorem 3.1 of [W1] will see interesting parallels in what follows.

4.2. *Lemma. Let* A,B *be disjoint closed subsets of a space* $\tilde{X}$, *let* S *be a separator of* (A,B) *in* $\tilde{X}$, *and let* U *be an open neighborhood of* S *in* $\tilde{X}$. *Then for each* 0-*dimensional subset* P *of* $\tilde{X}$ *there exists a separator* S* *of* (A,B) *in* $\tilde{X}$ *such that* $S^* \subset U - P$.

Proof. Let $\tilde{X} - S = V_1 \cup V_2$ where $A \subset V_1$, $B \subset V_2$, both V_1, V_2 are open and $\overline{V}_1 \cap V_2 = \emptyset = V_1 \cap \overline{V}_2$. Choose an open set W so that $S \subset W \subset U - (A \cup B)$. The sets $V_1 - W$ and $V_2 - W$ are closed in $\tilde{X}$; for example $V_1 - W = \tilde{X} - (W \cup V_2)$. Hence there exists a separator S* of $(V_1 - W, V_2 - W)$ in $\tilde{X}$ such that $S^* \cap P = \emptyset$. Since $A \subset V_1 - W$, $B \subset V_2 - W$ and $S^* \subset W \subset U$, the proof is complete.

4.3. *Lemma. Let* $K \subset \tilde{X}$ *be such that* $\dim K \leq m < \infty$. *Suppose* $\{(A_i', B_i') \mid 1 \leq i \leq n\}$ *is a collection of disjoint pairs of closed subsets of* $\tilde{X}$ *and that for each* i, S_i *is a separator of* (A_i', B_i') *in* $\tilde{X}$. *Let* U *be a neighborhood of* $S = \cap\{S_i \mid 1 \leq i \leq n\}$. *Then for each* i *there exists* S_i^* *so that* S_i^* *is a separator of* (A_i', B_i') *in* $\tilde{X}$, $S^* = \cap\{S_i^* \mid 1 \leq i \leq n\} \subset U$, *and* $\dim(S^* \cap K) \leq m - n$.

Proof. Write $K = \cup\{K_j \mid 1 \leq j \leq m + 1\}$ with the property that $\dim K_j \leq 0$ for each j. Let $S' = S_1 - U$ and let $S'' = \cap\{S_i - U \mid 2 \leq i \leq n\}$. Then $S' \cap S'' = \emptyset$ so there is an open set U' with $S' \subset U'$ and $U' \cap S'' = \emptyset$. Let $U_1 = U \cup U'$; hence $S_1 \subset U_1$. Use 4.2 to choose $S_1^* \subset U_1$ so that S_1^* is a separator of (A_1', B_1') and $S_1^* \cap K_1 = \emptyset$. We see that $S_1^* \cap S_2 \cap \cdots \cap S_n \subset U$ and $\dim(S_1^* \cap S_2 \cap \cdots \cap S_n \cap K) \leq m - 1$. This process can be repeated recursively to obtain the desired result.

Proof of 4.1. We shall find a closed subset A of X_2 and a map f of A to S^n that cannot be extended to a map of X_2 to K_n. Let $\Gamma = \{1,3,\cdots,2n+3\}$. Then Q_Γ is a copy of I^{n+2} which we choose to write in the form $I_1 \times I^{n+1}$. Let $\pi: Q \to I^{n+1} \subset I_1 \times I^{n+1}$ be the projection, and choose C to be a closed collar neighborhood of $S^n = \partial I^{n+1}$. Define A to be $X_2 \cap \pi^{-1}(C)$. The map f on A is given by π followed by a retraction ρ of C to S^n. Suppose f extends to a map $F: X_2 \to K_n$; then using the fact that K_n is an ANE, assume F is defined on a neighborhood V of X_2 in Q.

There exists r such that $\cap\{cl_Q(Z_k \cap Y) \mid k = 2,4,\cdots,2r\} \subset V$. Hence $[\cap\{Z_k \mid k = 2,4,\cdots,2r\}] \cap Y \subset V$. We may as well assume $2r > 2n + 3$, and we define Λ to be $\{1,2,\cdots,2r\}$. Let $U = V \cap Q_\Lambda$, and let $S_k = Z_k \cap Q_\Lambda$, $k = 2,4,\cdots,2r$. We have $[\cap\{Z_k | k = 2,4,\cdots,2r\}] \cap Y \cap Q_\Lambda = [\cap\{S_k | k = 2,4,\cdots,2r\}] \cap Y_\Lambda \subset V \cap Q_\Lambda = U$. Note that $\dim Y_\Lambda = 2r - 1$. Since $S_k \cap Y_\Lambda$ is a separator of $(A_k^t \cap Y, B_k^t \cap Y_\Lambda)$ in Y_Λ, use 4.3 to select separators S_k^* in Y_Λ so that $S^* \subset U$ and $\dim S^* \le r - 1$. If there exist odd numbers k so that $2n + 3 < k < 2r$, then for those k choose separators S_k^* of $(A_k^t \cap Y_\Lambda, B_k^t \cap Y_\Lambda)$ in Y_Λ so that $\dim S_0 \le n + 1$ where S_0 is the intersection of all S_k^*. Employing 3.1 and 2.5, $\{(A_j^t \cap S_0, B_j^t \cap S_0) | j = 3,5,\cdots,2n + 3\}$ is an essential family for S_0, so $\dim S_0 = n + 1$.

The map F restricted to S_0 is homotopic to a map $H: S_0 \to K_n$ that carries S_0 into the $(n+1)$-skeleton of K_n, i.e., into S^n itself. This map H can be chosen so that for some collar C_0 of S^n in I^{n+1}, $H = F$ on $\pi^{-1}(C_0) \cap S_0$. There exists $0 < s \le t$ so that $(A_j^s \cap S_0) \cup (B_j^s \cap S_0) \subset \pi^{-1}(C_0)$ for all $j = 3,5,\cdots,2n + 3$. The set $\{(A_j^s \cap S_0, B_j^s \cap S_0) | j = 3,5,\cdots,2n + 3\}$ is an essential family for S_0. The existence of the map H is contradictory to Proposition 4.3 of [W1]. For convenience, that Proposition is now stated (compactness in the hypothesis is unnecessary).

4.4. *Proposition. Let* X *be a space, let* $\{(A_k', B_k') | 1 \le k \le n + 1\}$ *be a family of disjoint pairs of closed subsets of* X, *and let* $f_k: X \to I_k$ *with* $A_k' = f_k^{-1}(-1)$ *and* $B_k' = f_k^{-1}(1)$. *The family* $\{(A_k', B_k') | 1 \le k \le n + 1\}$ *is essential if and only if the mapping* $f: X \to I^{n+1}$ *defined by* $f = (f_1, f_2, \cdots, f_{n+1})$ *is stable.*

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