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1. Introduction and Definitions

The purpose of this paper is to demonstrate a class of mappings \mathcal{J} , called simple folds, on trees such that every tree-like continuum has an inverse limit representation for which every bonding map is in the class \mathcal{J} . Furthermore, \mathcal{J} is, in a sense, the simplest such class. In terms of the structure of finite covers of tree-like continua, it could be said that this paper concerns the subject of "amalgamations." See for example [L] and [M].

A *continuum* is a compact connected metric space. A *tree* is a non-degenerate, connected, finite acyclic graph. If P is a point of the tree T , then $\text{order}(P) =$ the number of components of $T - \{P\}$. A *mapping* is a continuous function and is considered to be onto unless otherwise indicated. A mapping f of a tree T_1 onto a tree T_2 is called *p.l.* if and only if there exists a finite $V \subset T_1$ such that if C is a component of $T_1 - V$, then \bar{C} is an arc and $f|_{\bar{C}}$ is a homeomorphism. Two mappings, $f: X \rightarrow Y$ and $g: X \rightarrow Y$ are called *topologically equivalent* if there exist homeomorphisms $h: X \rightarrow X$ and $k: Y \rightarrow Y$ such that $kfh = g$.

If T is a tree, then the continuum M is *T-like* if and only if for each $\epsilon > 0$, there exists an ϵ -map $\pi: M \rightarrow T$. The continuum M is *tree-like* if and only if for each $\epsilon > 0$, there exists a tree T and an ϵ -map $\pi: M \rightarrow T$.

2. Simple Folds

Definition. Suppose that each of T_1 and T_2 is a tree, $P \in T_1$, T_a and T_b are subtrees of T_1 , $T_a \cup T_b = T_1$ and $T_a \cap T_b = \{P\}$. If $\beta: T_1 \rightarrow T_2$ is *not* a homeomorphism but each restriction $\beta|_{T_a}$ and $\beta|_{T_b}$ is a homeomorphism, then β is called a *fold*. If furthermore, one of the trees T_a or T_b is the closure of a component of $T_1 - \{P\}$, then β is called a *simple fold*.

We make the following observations concerning the foregoing definitions:

obs. 1) If $\beta: T_1 \rightarrow T_2$ is p.l., then β is a fold if and only if there is only one point of T_1 at which β is not locally 1-1.

obs. 2) If $\beta: T_1 \rightarrow T_2$ is a fold and $\text{order}(P) = 2$, then β is a simple fold.

obs. 3) If $\beta: T_1 \rightarrow T_2$ is a simple fold, then on an open set containing P , β identifies two components and only two components of $T_1 \cap O \sim \{P\}$ and $\text{order}(\beta(P)) = \text{order}(P) - 1$.

The examples in diagram 1 serve to illustrate some of the types of folds. The mappings β_1 and β_2 are the "simplest" of the simple folds and will be referred to again later. The mapping β_1 is a simple fold of a triod onto an arc and β_2 is a simple fold of an arc onto a triod. With the proper choice of T_a and T_b , β_3 is a fold but cannot be a simple fold. β_4 is a simple fold.

3. The Factorization Theorem

In this section we will prove a general theorem about the factorization of mappings by folds and then make some

observations based upon the proof.

Theorem 1. Suppose each of T_1 and T_2 is a tree and f is a light mapping of T_1 onto T_2 which is not a homeomorphism. Then there exists a tree T_3 , a map $\alpha: T_1 \rightarrow T_3$ and a fold $\beta: T_3 \rightarrow T_2$ such that $f = \beta\alpha$.

Proof. Let a and b be two points of T_1 such that $f(a) = f(b)$. Since f is light, there exists a point P which separates a from b in T_1 and such that $f(P) \neq f(a)$. Let T_a and T_b be two trees which contain the points a and b respectively and such that $T_a \cup T_b = T_1$ and $T_a \cap T_b = \{P\}$. Now each of $f(T_a)$ and $f(T_b)$ is a nondegenerate subtree of T_2 and $f(T_a) \cup f(T_b) = T_2$.

We construct the tree T_3 as the union of a homeomorph of $f(T_a)$ and a homeomorph of $f(T_b)$ joined only at the point $f(P)$. Thus we define homeomorphisms β_a and β_b such that $\beta_a: f(T_a) \rightarrow \beta_a f(T_a) \subset T_3$ and $\beta_b: f(T_b) \rightarrow \beta_b f(T_b) \subset T_3$, $\beta_a f(T_a) \cup \beta_b f(T_b) = T_3$ and $\beta_a f(T_a) \cap \beta_b f(T_b) = \{\beta_a f(P) = \beta_b f(P)\}$.

The mapping $\alpha: T_1 \rightarrow T_3$ is defined by

$$\alpha(x) = \begin{cases} \beta_a f(x) & \text{for } x \in T_a \\ \beta_b f(x) & \text{for } x \in T_b \end{cases}$$

The continuity of α is assured since $T_a \cap T_b = \{P\}$ and $\beta_a f(P) = \beta_b f(P)$. The mapping $\beta: T_3 \rightarrow T_2$ is defined by

$$\beta(x) = \begin{cases} \beta_a^{-1}(x) & \text{for } x \in \beta_a f(T_a) \\ \beta_b^{-1}(x) & \text{for } x \in \beta_b f(T_b) \end{cases}$$

Likewise, β is continuous since $\beta_a f(T_a) \cap \beta_b f(T_b) = \{\beta_a f(P) = \beta_b f(P)\}$.

It is easy to check now that $\beta\alpha(x) = f(x)$ for all $x \in T_1$. It remains now to show that β is a fold. β is obviously the union of two homeomorphisms joined at a point.

Now consider the points $\alpha(a), \alpha(b) \in T_3$. We have $\alpha(a) = \beta_a f(a) \in \beta_a f(T_a)$ and $\alpha(b) = \beta_b f(b) \in \beta_b f(T_b)$. It is not possible for $\alpha(a) = \alpha(b)$ since this would imply that $\alpha(a) = \beta_a f(P)$ which is the only point in $\beta_a f(T_a) \cap \beta_b f(T_b)$. If so, then $f(a) = \beta\alpha(a) = \beta \beta_a f(P) = f(P)$ contrary to the way that P was chosen.

We now have $\alpha(a) \neq \alpha(b)$ but $\beta\alpha(a) = \beta\alpha(b)$. Since β is not a homeomorphism, β satisfies the definition of a fold and the proof is complete.

The proof of Theorem 1 is really quite simple. The mapping α splits the image of f into two parts and the mapping β mends the parts back together. It is hoped, however, that the more detailed argument given will facilitate our making a few more observations based upon the proof.

obs. 4) The mapping α is a homeomorphism if and only if each of $f|_{T_a}$ and $f|_{T_b}$ is a homeomorphism.

obs. 5) If f is p.l., then each of α and β is p.l.

obs. 6) Let $[a,b]$ denote the subarc of T_1 joining a and b . The point P could have been chosen so that $f(P)$ is an end point of the subtree $f([a,b])$ and $f(P)$ is different from $f(a)$. In such a case, f is not locally 1-1 at P .

obs. 7) If $\text{order}(P) = 2$ and f is *not* locally 1-1 at P , then α is locally 1-1 at P .

obs. 8) If f is a fold, then f is a finite composition of simple folds. To prove this, we choose P to be the

unique point at which f is not locally 1-1 and apply Theorem 1 to obtain $f = \beta\alpha$. In this case, α will be a fold and so if either α or β is not a simple fold, then Theorem 1 can be applied again to α using the point P or to β using the point $\alpha(P)$ and so on. If f locally maps m arcs onto n arcs, then each simple fold factor merely splits one identification. The result will be a composition of $m - n$ simple folds.

We now have a slight improvement upon Theorem 1.

Theorem 2. Suppose each of T_1 and T_2 is a tree and f is a light mapping of T_1 onto T_2 which is not a homeomorphism. Then there exist a tree T_3 , a map $\alpha: T_1 \rightarrow T_3$ and a simple fold $\beta: T_3 \rightarrow T_2$ such that $f = \beta\alpha$.

Proof. The proof follows from Theorem 1 and obs. 8.

Theorem 3. If f is a p.l. mapping of a tree onto a tree then f is either a homeomorphism or is a finite composition of simple folds.

Proof. We will first show that f is a finite composition of folds. If $f: T_1 \rightarrow T_2$ is p.l. and not a homeomorphism, then we choose a point P of T_1 at which f is not locally 1-1 (as in obs. 6) and apply Theorem 1 to obtain $f = \beta\alpha$ where β is a fold and α is p.l. (obs. 5). If α is a homeomorphism, then $f = \beta\alpha$ is a fold. And if α is not a homeomorphism, then one of two cases must occur: First, α performs at least one identification of a pair of subarcs in a neighborhood of P but fewer such identifications than f does, or second, α is locally 1-1 at P but fails to be locally 1-1 at some other point.

In the first case we can choose the point P again and apply Theorem 1 to the map α . In the second case, we choose another point $Q \neq P$ such that α fails to be locally 1-1 at Q and apply Theorem 1. We obtain $\alpha = \beta'\alpha'$ where β' is a fold and α' is p.l. We apply Theorem 1 to α' and continue in this manner.

Since f fails to be 1-1 at only finitely many points and makes only finitely many identifications at each such point, this process must terminate. Now that f is a composition of finitely many folds, we can factor each fold into finitely many simple folds (obs. 8). This completes the proof.

Each simple fold in the factorization of f can be thought of as performing one of the local identifications done by f . The factorization obtained will only depend upon the order in which the identifications are to be removed. Thus if f performs n local identifications, then the number of possible factorizations of f into simple folds would seem to be $n!$ In general, however, some duplications will occur.

It should also be pointed out that simple folds are not "indivisible." The simple fold β_2 (see diagram 1) is the composition of two simple folds. Let $\beta_2 = \beta_2''\beta_2'$ where β_2' is topologically equivalent to β_2 and β_2'' identifies parts of the two bottom legs of the triod.

4. Applications

In this section we give some applications to inverse limit spaces. We begin with a corollary to Theorem 3.

Corollary 1. If f is a map of a tree T_1 onto a tree T_2 and $\epsilon > 0$, then there exist simple folds $\beta_1, \beta_2, \dots, \beta_n$ such that $d(f, \beta_1 \beta_2 \dots \beta_n) < \epsilon$.

Proof. The proof follows from Theorem 3 and the fact that any map can be uniformly approximately by p.l. maps.

Corollary 2. If M is a tree-like continuum, then M is the limit of an inverse limit system $T_1 \xleftarrow{\beta_1} T_2 \xleftarrow{\beta_2} T_3 \xleftarrow{\beta_3} \dots M$ where each of the bonding maps is a simple fold.

Proof. It follows from the basic theory contained in [M-S], the approximation theorem of [B] and Corollary 1 that every tree-like continuum M has an inverse limit representation $T_1 \xleftarrow{f_1} T_2 \xleftarrow{f_2} T_3 \xleftarrow{f_3} \dots M$ where each bonding map is a finite composition of simple folds. The inverse limit space is the same after taking as bonding maps the individual simple fold factors.

Even though the bonding maps in the representation given by Corollary 2 are relatively simple, there is not, in general, any control over the complexity of the trees which appear as factor spaces. Of course the number of endpoints of the trees must be unbounded if M is infinitely branched [Y₂]. In case M is arc-like, we can offer the following:

Theorem 4. There exists a simple fold $\beta_1: \text{triod} \rightarrow \text{arc}$ and a simple fold $\beta_2: \text{arc} \rightarrow \text{triod}$ such that if M is an arc-like continuum, then M is the inverse limit of the inverse limit system $T_1 \xleftarrow{\beta_1} T_2 \xleftarrow{\beta_2} T_3 \xleftarrow{\beta_3} \dots M$ such that for each $i \geq 1$, β_{2i} is topologically equivalent to β_2 and β_{2i+1} is topologically equivalent to β_1 .

Proof. The mappings β_1 and β_2 are the same as in diagram 1.

In Corollary 3.7 of [Y₁] it is shown that there are two maps of the unit interval onto itself, L and Z such that every map of the unit interval onto itself can be uniformly approximated by a composition of maps each of which is either L or Z or a homeomorphism. In other words, every map of the unit interval onto itself can be uniformly approximated by a composition of maps each of which is topologically equivalent to either L or Z. See diagram 2 for a description of the maps L and Z.

As was pointed out in the proof of Corollary 2, it follows that every arc-like continuum M has an inverse limit representation

$$\text{arc} \xleftarrow{f_1} \text{arc} \xleftarrow{f_2} \text{arc} \xleftarrow{f_3} \dots M$$

where each bonding map is topologically equivalent to either L or Z. Now the map Z is the composition of β_2 followed by β_1 if β_1 is carefully applied so that the leg of the triod which receives the interior point of the arc is folded down into one of the other two legs. The map L is also the composition of β_2 followed by β_1 . Take one of the legs of the triod which receives an endpoint of the arc and apply β_1 to fold that leg into the leg which received the other end point.

So if M is an arc-like continuum, then M has an inverse limit representation

$$\text{arc} \xleftarrow{\beta_1} \text{trioid} \xleftarrow{\beta_2} \text{arc} \xleftarrow{\beta_3} \text{trioid} \leftarrow \dots M$$

where for each i, β_{2i+1} is topologically equivalent to β_1 and β_{2i} is topologically equivalent to β_2 . This completes the proof.

DIAGRAM I

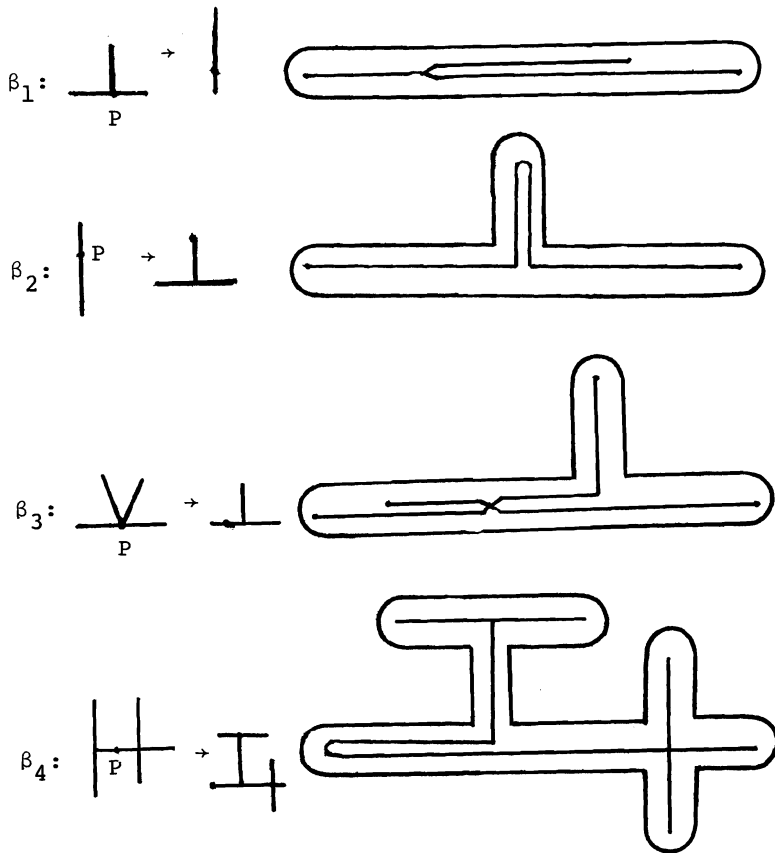
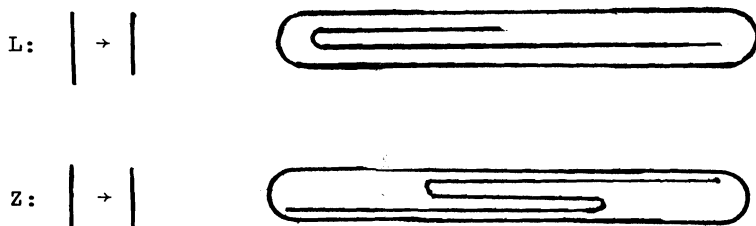


DIAGRAM II



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