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The purpose of this note is to give some characterizations by means of a special kind of normality of certain product topological spaces as well as by extension of continuous functions with values in normed linear spaces.

Let us recall some notations and definitions.

All topological spaces are assumed to be completely regular Hausdorff spaces. The topological space and its support are denoted with the same letter; for any subset A of a topological space X, $\overline{A}$ stands for the closure of A.

For any set Y let A(Y) denote the Alexandrov's (= one point) compactification of the discrete space of support Y.

Definition 1. A topological space X is strongly collectionwise Hausdorff if for any closed discrete subset F of X, there is a discrete family of open subsets of X, $(U_x)_{x \in F}$, such that $x \in U_x$, $\forall x \in F$.

Remark. A normal collectionwise Hausdorff space is strongly collectionwise Hausdorff.

Definition 2. A space X is discrete normal if any two disjoint closed subsets of X, one of which is discrete, can be separated by open sets.

Definition 3. A space X is strongly collectionwise normal for compact sets if for any discrete family of compact subsets of X, $(F_i)_{i\in I}$, there is a discrete family of open subsets of X, $(U_i)_{i\in I}$, such that $F_i \subset U_i$, $\forall i \in I$.

Definition 4. A space X is compact normal if any two disjoint closed subsets of X, one of which is the union of a discrete family of compact subsets of X, can be separated by open sets.

Remark. It is immediate that strong collectionwise Hausdorffness implies discrete normality and that strong collectionwise normality for compact sets implies compact normality.

Theorem 1. *Let* X *be a strongly collectionwise normal for compact sets space. If* $\mathcal{F}$ *is a locally finite collection of compact subsets of* X, *there is a locally finite (in its union) collection of open subsets of* X, $\{U_F \mid F \in \mathcal{F}\}$, *such that* $F \subset U_F$, $\forall F \in \mathcal{F}$.

Proof. Since $\mathcal{F}$ is a locally finite collection of compact subsets of X, there is a sequence (D_n) of pairwise disjoint subsets of $\mathcal{F}$, whose union is $\mathcal{F}$, and such that each D_n is a discrete collection in X.

Let U_F denote the open set assigned to each $F \in \mathcal{F}$. Choose a discrete collection $\{U_F \mid F \in D_n\}$ by induction on n so that

1) $F \cap K = \emptyset$ implies $\overline{U}_F \cap K = \emptyset$' whenever $F \in D_n$ and $K \in \mathcal{F}$;

2) $F \cap K = \emptyset$ implies $U_F \cap U_K = \emptyset$ whenever $F \in D_n$ and $K \in D_1 \cup \cdots \cup D_n$.

The collection $\{U_F \mid F \in \mathcal{F}\}$ is as required.

Theorem 2. *If* X *is a strongly collectionwise Hausdorff (respectively, countably paracompact, strongly collectionwise normal for compact sets) space, then the topological product* X × K *is strongly collectionwise Hausdorff (respectively, strongly collectionwise normal for compact sets), for any compact space* K.

Hint. The projection from X × K onto X is closed and the image collection under this projection of a discrete collection of compact sets is a locally finite collection of compact subsets of X.

Remark. Let $N = \{1,2,3\cdots\}$ with the discrete topology, X be topological space and $\mathcal{J}$ be a σ-discrete collection of compact subsets of X--say $\mathcal{J} = \cup\{D_n | n \in N\}$. Then $\{F \times \{n\} | F \in D_n | n \in N\}$ is a discrete collection of compact sets in the product space X × A(N). It thus follows that X × A(N) compact normal and X strongly collectionwise normal for compact sets implies X × A(X) is compact normal.

Proceeding as in [1] we have

Theorem 3. *A space* X *is strongly collectionwise Hausdorff if and only if* X × A(X) *is discrete normal. A countably paracompact space* X *is strongly collectionwise normal for compact sets if and only if* X × A(X) *is compact normal.*

Theorem 4. *Let* X *be a strongly collectionwise Hausdorff (respectively, strongly collectionwise normal for compact sets) space,* B *be a normed linear space,* F *be a closed* *discrete (respectively, union of a discrete family of compact subsets of* X*) and* f: F → B *be a continuous function. Then* f *can be continuously extended to* X.

Proof. Let K be a compact subset of X and g: K → B be a continuous function; then the image set g(K) is a compact subset of the normed linear space B, so is homeomorphic to a compact subset of a convenient R^n. (This result follows from [6], page 75, Corollary and the characterization of the compact subsets of the Banach space of bounded sequences of real numbers with the sup norm.) From 3.2.J and 3.11.15 of R. Engelking's book General Topology (PWN-Polish Scientific Publishers, 1977) it follows that g is continuously extendable over X.

Let us now consider the general case. Let F be the union of a discrete family of compact subsets of X--say $(F_i)_{i \in I}$--eventually the F_i may be unitary sets. There is a discrete family of open subsets of X, $(U_i)_{i \in I}$, with $F_i \subset U_i$ for every $i \in I$. For each $i \in I$ fix a continuous function h_i: X → [0,1] so that $h_i(F_i) = \{1\}$ and h_i equal 0 outside U_i; furthermore, let g_i denote a continuous function from X into B which extends the restriction of f to F_i. Finally, we have that the function $\Sigma_{i \in I} h_i \cdot g_i$ is as required.

Definition 5. A subset A of a topological space X is P-embedded if every continuous pseudo-metric defined on A is continuously extendable on X.

Theorem 5. *A space* X *is strongly collectionwise Hausdorff (respectively, strongly collectionwise normal for* *compact sets) if and only if every closed discrete (respectively, union of a discrete family of compact subsets) set is* P*-embedded.*

Proof. It follows from [8].

Remarks. 1) By theorem 9 of [11] it is independent of the set theory axioms whether "there is a locally compact normal space which is not collectionwise Hausdorff." Under MA + $\neg$CH there is such space K and thus K × A(K) is not discrete normal.

2) In [5] Fleissner proves that under V = L, if X is normal space of character $\leq\aleph_1$, then it is collectionwise Hausdorff. But in the proofs he just uses discrete normality (instead of normality) so, as a matter of fact, he proves that, under V = L, if X is a discrete normal space of character $\leq\aleph_1$, then it is strongly collectionwise normal.

3) As a consequence of theorem 1 if X is a countably paracompact, locally compact, strongly collectionwise for compact sets space, then if F and G are disjoint closed subsets of X, one of which is paracompact, there is a continuous function f from X into [0,1] such that f(F) = {0} and f(G) = {1}.

4) It is interesting to compare theorem 3 and theorem 2 of [1], where a similar characterization for collectionwise normality is given. Since there are (countably paracompact) normal, collectionwise Hausdorff, noncollectionwise normal spaces (see, for instance, [10]), we have spaces X such that X × A(X) is not normal but it is discrete normal.

5) W. S. Watson has recently proved that there is a normal collectionwise Hausdorff space which is not collectionwise normal with respect to copies of [0,1].

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