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by

Mahendra Jani

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Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
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CELL-LIKE SHAPE FIBRATIONS WHICH ARE FIBER SHAPE EQUIVALENCES

Mahendra Jani[1]

I. Introduction

Cell-like mappings have been important in the study of mappings on manifolds. In some sense they generalize homeomorphisms and have certain approximate lifting properties [1,6]. Motivated by these facts D. Coram and C. Duvall [2] introduced approximate fibrations to study mappings with non-trivial point-inverses. "Approximate fibrations" is a right tool to study maps between ANR's but it misses some cannonical projections from local pathological compact metric spaces to ANR's. Also, it fails to have the simple property of being preserved under the pull-backs. However, it motivated S. Mardešić and T. B. Rushing [7] to generalize it to a notion called "shape fibration."

These facts guided the author of this paper to study "shape fibrations" in order to study bundle theory for locally pathological spaces. The first question one asks "are all the Dold-like theorems true for shape fibrations?" The answer is "yes" if we have a "right" notion of "fiber shape equivalence." In [4,5] the author has defined the notion and has proved a Dold-like theorem for shape fibrations. In this paper we are proving one of the important theorems for such maps. The question is the following:

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Suppose we are given two shape fibrations $p: E' \to B$ and $q: E \to B$ and a shape map $f: E' \to E$ over B. Also suppose we are given that the restriction of f to each fiber, $f|F_b$, $(b \in B)$ is a shape equivalence, then under what conditions (on B) is f a fiber shape equivalence?

If the shape fibration $q = 1_B: B \to B$, then the question reduces to the Question-3 of T. B. Rushing [10]: Under what conditions is a cell-like shape fibration is a shape equivalence?

Edwards and Hastings [3] give an example of a cell-like shape fibration which fails to be a shape equivalence. This justifies the Question-3 of [10].

In this paper we have found the condition as a new notion called "shrinkable open cover" and have proved the following theorem:

Theorem 1. *Let* $f: p \to q$ *be a shape map between two shape fibrations* $p: E' \to B$ *and* $q: E \to B$ *over* B *where* p *is movable. If* B *admits a shrinkable open cover then* f *is a fiber shape equivalence if and only if* $f|p^{-1}(b)$ *is a shape equivalence for a point* b *in each strong shape path component of* B.

Corollary 1. *If a compact metric space* B *admits a shrinkable open cover then every cell-like shape fibration* $p: E \to B$ *from a compactum* E *onto* B *is a fiber shape equivalence.*

The base space of the Edwards-Hastings example does not admit such an open cover.

By combining the above corollary with the results of [5] and [9], we obtain the following corollary which answers Question-3 of T. B. Rushing [10].

Corollary 2. *If a compact metric space* B *admits a shrinkable open cover then for a map* p: E $\to$ B *from a compactum* E, *the following three notions are equivalent.*

(1) *cell-like shape fibration*

(2) *fiber shape equivalence*

(3) *hereditary shape equivalence.*

Note that T. C. McMillan [9] has proved that in finite dimensions cell-like map, cell-like shape fibrations and hereditary shape equivalences are equivalent.

Hence the interest lies in determining the relationship in infinite dimensions.

In section II, we recall useful definitions and theorems [4,5], in section III, we define the new notion "shrinkable open cover" and show that strong FANR's admit such open covers while the dyadic solenoid and infinite product of spheres do not admit such open covers. In section IV, we prove the Theorem-1.

I thank Professor Dyer and Professor Hastings for inspiring conversations.

I thank the referee for suggesting a short name "shrinkable open cover" rather than a long "strongly shape trivial open cover"!

II. Preliminaries

All spaces considered will be compact metric spaces unless otherwise stated. By an ANR, we mean an absolute neighborhood retract for metric spaces.

An (open) ANR-sequence $\underline{B} = (B_n, \delta_{nm})$ is an inverse sequence of compact (open) ANR's. A *map* $\underline{f} = (f_n, \emptyset): \underline{E} = (E_n, \gamma_{nm}) \to \underline{B}$ of ANR-sequences is a sequence of maps $f_n: E_{\emptyset(n)} \to B_n$, $n = 1,2,3\cdots$ such that for $m \geq n$, $f_n\gamma_{nm} \simeq \delta nmfm: E_{\phi(m)} \times I \to B_n$. If the non-decreasing index function $\emptyset = 1_N$ and for $m \geq n$, $f_n\gamma_{nm} = \sigma_{nm}f_m$ then we refer $\underline{f}$ as a *level map* of ANR-sequences.

Two maps $\underline{f}$ and $\underline{g} = (g_n, \psi): \underline{E} \to \underline{B}$ of ANR-sequences are said to be *homotopic* ($\underline{f} \simeq \underline{g}$) if for every n, there is an index $\hat{n}(\underline{f},\underline{g}) \geq n$ such that for all $m \geq n$, there is a homotopy $H^{nm}: f_n\gamma_{\phi(n)m} \simeq g_n\gamma_{\psi(n)m}$. In addition, if for $\ell > n$ and $w > \ell$ there is a homotopy $H^{\ell w}: f_\ell\gamma_{\phi(\ell)w} \simeq g_\ell\gamma_{\psi(\ell)w}$ and these two homotopies are coherent (rel. 0,1), meaning there is an index $s \geq \max(m,w)$ and homotopy $H: E_s \times I \times I \to B_n$ such that $H(x,t,o) = H^{nm}(\gamma_{nw} \times 1_I)$; $H(x,t,1) = \delta_{n\ell}H^{\ell w}$ and $H(x,o,t) = f_n\gamma_{\phi(n)w}$ and $H(x,1,t) = \delta_{n\ell}g_\ell\gamma_{\psi(\ell)w}$, then $\underline{f}$ is said to be *strongly homotopic* to $\underline{g}$. Two maps (continuous functions) $f,g: E \to B$ between spaces are *strongly shape homotopic* if there are strongly homotopic level maps $\underline{f},\underline{g}: \underline{E} \to \underline{B}$ of ANR-sequences with limit maps f and g.

A *map* $\underline{f}$: $\underline{p} \to \underline{q}$ *between two level maps* $\underline{p}: \underline{E}' \to \underline{B}$ and $\underline{q}: \underline{E} \to \underline{B}$ over $\underline{B}$ is a map $\underline{f}: \underline{E}' \to \underline{E}$ of ANR-sequences such that for any n and any $\varepsilon > 0$ there is an index $n^*(\underline{f},\varepsilon) \geq n$ satisfying the following conditions:

(i) $d(\delta_{nm}q_m f_m, \delta_{nm}p_m) < \varepsilon$ for all $m \geq n^*$ $(\underline{f},\varepsilon)$ and

(ii) for all $\ell \geq m \geq n^*$ $(\underline{f},\varepsilon)$ there is a homotopy $H^{m\ell}$: $f_m\gamma'_{\phi(m)\phi(\ell)} \simeq \gamma_{m\ell}f_\ell$: $E'_{\phi(\ell)} \times I \to E_m$ such that $d(\delta_{nm}q_{n\ell}H^{m\ell}, \delta_{n\ell}p_\ell)$ for every $t \in I$. We refer $M^{m\ell}$ as an ε-vertical homotopy.

Two maps $\underline{f},\underline{g}$: $\underline{p} \to \underline{q}$ over $\underline{B}$ between two level maps of ANR-sequences are said to be *fiber homotopic* ($\underline{f} \underset{\underline{B}}{\simeq} \underline{g}$) if for every n and for every $\varepsilon > 0$ there is an index n^* $(\underline{f},\underline{g};\varepsilon) \geq n$ such that for all $m \geq n^*$ $(\underline{f},\underline{g};\varepsilon)$ there is an ε-vertical homotopy G^{nm}: $f_n\gamma'_{\phi(n)m} \simeq g_n\gamma'_{\psi(n)m}$.

Two level maps $\underline{p}$: $\underline{E}' \to \underline{B}$ and $\underline{q}$: $\underline{E} \to \underline{B}$ are said to be *fiber homotopy equivalent* ($\underline{p} \simeq \underline{q}$) if there are fiber maps $\underline{f}$: $\underline{p} \to \underline{q}$ and $\underline{g}$: $\underline{q} \to \underline{p}$ over $\underline{B}$ such that $\underline{gf} \underset{\underline{B}}{\simeq} \underline{1}_{\underline{p}}$ and $\underline{fg} \underset{\underline{B}}{\simeq} \underline{1}_{\underline{q}}$.

For given two maps p: E' → B and q: E → B between spaces, a shape map f: p → q over B is an equivalence class $[\underline{f}]$ of a fiber map $\underline{f}$: $\underline{p} \to \underline{q}$ between two level maps over $\underline{B}$ with $\varprojlim \underline{p} = p$ and $\varprojlim \underline{q} = q$.

The *shape map* $f = [\underline{f}]$: $p \to q$ *over* $\underline{B}$ is a fiber shape equivalence if there is a shape map $g = [\underline{g}]$: $q \to p$ over B such that $\underline{gf} \underset{\underline{B}}{\simeq} \underline{1}_{\underline{p}}$ and $\underline{fg} \underset{\underline{B}}{\simeq} \underline{1}_{\underline{q}}$.

Two shape fibrations p: E' → B and q: E → B are *fiber shape equivalent* if there is a fiber shape equivalence f: p → q.

Now, we recall the definition of an "open set" of an ANR-sequence.

An "*open set*" $\underline{u} = (u_n)$ *of an* ANR-*sequence* $\underline{B} = (B_n\delta_{nm})$ is a sequence of open sets u_n, n = 1,2,3,··· such that for each n, u_n is an open set of B_n and for $m \geq n$, $\delta_{nm}(\bar{u}_m) \subset u_n$ where $\bar{u}_m$ is the closure of u_m.

A collection of open covers $\underline{\mathcal{U}} = \{\mathcal{U}_n\} n = 1,2,\cdots$ is called an "open cover" of $\underline{B} = (B_n, \delta_{nm})$ if

(i) for each $n = 1,2,\cdots$, $\mathcal{U}_n$ is an open cover of B_n and

(ii) there are "open sets" $\underline{u}^j = (u_n^j)$ of $\underline{B}$ such that $B = \cup_{j \in J}\{\text{int.}(\varprojlim \underline{u}^j = (u_n^j))\}$ where $\varprojlim \underline{B} = B$.

If the index set J is finite then we say that the "open cover" $\underline{\mathcal{U}}$ is finite. The "open set" $\underline{u}^j$ of $\underline{B}$ is said to be associated with the open set $u^j = \text{int}(\varprojlim \underline{u}^j)$ of B.

Remark 1. For a given finite open cover $\mathcal{U}$ of a space B there is a "finite open cover" $\underline{\mathcal{U}}$ of $\underline{B}$ associated with $\mathcal{U}$ where $\varprojlim \underline{B} = B$.

Remark 2. By the definition of the "open set" $\underline{u}$ of $\underline{B}$, it is clear that $\varprojlim \underline{u}^j = \lim \underline{\bar{u}}^j$ where $\underline{\bar{u}}^j = (\bar{u}_n^j | \bar{u}_n^j$ is the closure of the open set u_n^j for each $n = 1,2,\cdots)$.

The author has defined the movability condition for a map in [5] as follows:

A level map $\underline{P}$: $\underline{E} \to \underline{B}$ of ANR-sequences is said to be *vertically strongly movable* if for every index n and $\varepsilon > o$, there is an index $m \geq n$ such that for any $\ell \geq m$ there is a map $\eta_{\ell m}: E_m \to E_\ell$ satisfying the following conditions:

(1) $\eta_{\ell m}\gamma_{m\ell} = 1_{E_\ell}$ and

(2) there is an ε-vertical homotopy H: $\gamma_{nm}\gamma_{m\ell}\eta_{\ell m} \underset{B_n}{\tilde{\simeq}} \gamma_{nm}$: $E_m \to E_n$ (rel. $\gamma_{m\ell}(E_\ell)$).

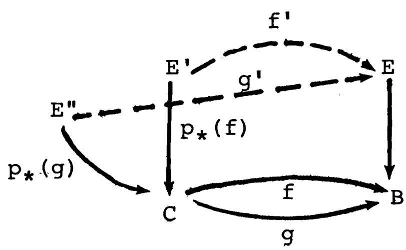
A level map $\underline{p}$: $\underline{E} \to \underline{B}$ with $\varprojlim \underline{p} = p$ is said to be *movable* if for every open cover $\mathcal{U}$ of $\underline{B}$ there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that for each open set $\underline{V}$ of $\mathcal{V}$, the restriction map $\underline{p}(\underline{v})$: $\underline{p}^{-1}(\underline{v}) \to \underline{v}$ is vertically strongly movable.

A map p: E $\to$ B between compact metric spaces is movable if every level map $\underline{p}$: $\underline{E} \to \underline{B}$ with $\varprojlim \underline{p} = P$ is movable.

Now we will recall the useful theorems from [4,5] (Theorem-A has also been proved independently by A. Matsumoto [8]):

Theorem A. *The pull-back of a shape fibration is a shape fibration.*

Theorem B. *Two strongly shape homotipic maps induce fiber shape equivalent shape fibrations, that is if* $(E';p_*(f),f')$



and $(E";p_*(g),g')$ *are pull-back diagrams of spaces then* $f \simeq g \Rightarrow p_*(f) \cong p_*(g)$.

Two points x and y of a space B are said to be connected by a strong path if the two inclusion maps i_x, i_y: $* \hookrightarrow$ B are strongly shape homotopic.

Corollary B. *For a shape fibration* p: E $\to$ B, *all the fibers over the points in the same strong shape path component of* B *are of the same shape.*

Theorem C. *Let* $p: E^1 \to B$ *and* $q: E \to B$ *be shape fibrations where* p *is movable and* $\underline{p}: \underline{E}' \to \underline{B}$ *and* $\underline{q}: \underline{E} \to \underline{B}$ *be level maps of* ANR-*sequences with* $\varprojlim \underline{p} = p$ *and* $\varprojlim \underline{q} = q$. *Let* $\underline{f}: \underline{E}' \to \underline{E}$ *be a map over* $\underline{B}$ *such that* $\underline{f}|\underline{p}^{-1}(\bar{\underline{u}})$ *is a fiber homotopy equivalence for each* $\bar{\underline{u}}$ *where* $\bar{\underline{u}}$ *is the "closure" of the "open set"* $\underline{u}$ *of* $\underline{B}$, *then* $\underline{f}$ *is a fiber homotopy equivalence.*

Note that the theorem C has been proved for $\underline{u}$ but the proof also works out for $\bar{\underline{u}}$.

For more details the reader is advised to refer [4,5].

III. Shrinkable Open Cover

In this section we define the notion of a "shrinkable open cover" and find the class of spaces that admit such an open cover.

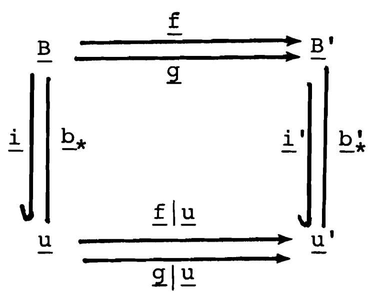
Definition 1. A non-empty "open set" $\underline{u}$ of an ANR-sequence $\underline{B} = (B_n, \delta_{nm})$ is said to be "storngly homotopy trivial" if the inclusion level map $\underline{i}: \bar{\underline{u}} \hookrightarrow \underline{B}$ of ANR-sequences from the "closure" of $\underline{u}$ to $\underline{B}$ is strongly homotopic to the constant level map $\underline{b}_*: \bar{\underline{u}} \to \underline{B}$ where $\underline{b}' = (b_n | b_n \in B_n$ for each $n = 1,2,3,\cdots)$ is a "point" in $\underline{B}$ (for $m \geq n$, $\delta_{nm}(b_m) = b_n$).

Definition 2. An ANR-sequence $\underline{B}$ is said to admit a "shrinkable open cover" $\underline{\mathcal{U}}$ if there is an "open cover" $\underline{\mathcal{U}} = \{\mathcal{U}_n\}$ of $\underline{B}$ such that each "open set" $\underline{u}$ of $\underline{\mathcal{U}}$ is strongly homotopy trivial.

The following proposition-1 shows that the admittance of the strongly homotopy trivial open cover is invariant under the choice of an ANR-sequence.

Proposition-1. *Let* $\underline{B}$ *and* $\underline{B}'$ *be* ANR-*sequences with* $\varprojlim \underline{B} = \varprojlim \underline{B}' = B$. *If* $\underline{\mathcal{U}}$ *is a "shrinkable finite open cover" of* $\underline{B}$ *associated with a finite open cover* $\mathcal{U}$ *of* B *then there is a "shrinkable open cover"* $\underline{\mathcal{U}}'$ *of* $\underline{B}'$ *or of its confinal subsequence* $\underline{B}'$, *associated with* $\mathcal{U}$.

Proof. Since $\varprojlim \underline{B}' = B$, there are maps of ANR-sequences $\underline{f}: \underline{B} \to \underline{B}'$ and $\underline{g}: \underline{B}' \to \underline{B}$ ([4], Lemma-1) such that $\underline{g}\underline{f}$ and $\underline{f}\underline{g}$ are strongly homotopic to $\underline{1}_{\underline{B}}$ and $\underline{1}_{\underline{B}'}$, respectively. Select "finite open cover" $\underline{\mathcal{U}}'$ of $\underline{B}'$ associated with the open cover $\mathcal{U}$ of B such that for each "open set" $\underline{u}'$ of $\mathcal{U}'$, $\underline{g}(\overline{\underline{u}'}) \subset \underline{u}$ where $\underline{u}'$ and $\underline{u}$ are associated with an open set u of $\mathcal{U}$. Consider the diagram-2.



Now, $\underline{i}' = \underline{1}_{\underline{B}'}\underline{i}' \underset{(1)}{\simeq} \underline{f}\underline{g}\underline{i}' = \underline{f}\underline{i}\underline{g}_{\underline{u}'} \underset{(2)}{\simeq} \underline{f}\underline{b}_*\underline{g}_{\underline{u}'} \underset{(3)}{\simeq} \underline{f}\underline{g}\underline{b}'_* \underset{(4)}{\simeq} \underline{1}_{\underline{B}'}\underline{b}'_* = \underline{b}_*$

Where (1) and

(2) are given

(3) is by the construction of the map $\underline{g}$

(4) homotopies are induced by $\underline{f}\underline{g} \simeq \underline{1}_{\underline{B}'}$

Also, each homotopy can be made as small as we wish by selecting numbers ε's and indices.

Coherent conditions on homopies of $\underline{i} \simeq \underline{b}_*$ induce coherent conditions on $\underline{i}' \simeq \underline{b}_*$. The process is straightforward but technically complicated. So we omit it here.

Definition 3. A compact metric space B is said to admit a shrinkable open cover if there is an ANR-sequence $\underline{B}$ with $\varprojlim \underline{B} = B$ that admits a shrinkable open cover associated with the open cover of B.

Remark-3. Since B is a compact metric space in the definition-3, if B admits a shrinkable open cover then it admits a shrinkable finite open cover.

We will find the class of spaces admitting such open covers.

A space B is said to admit homotopy trivial open cover $\mathcal{U}$ if for each non-empty open set u of $\mathcal{U}$, the inclusion map $i: u \hookrightarrow B$ is homotopic to the constant map $b_* = u \to B$.

Proposition-2. *If a space admits a (finite) homotopy trivial open cover then it admits a shrinkable (finite) open cover.*

Proof. Suppose a space B which admits finite homotopy trivial open cover is embedded in the Hilbert cube $\mathbf{Q}$. Select decreasing sequence of compact ANR neighborhoods B_n of B such that $\cap_n B_n = B$. Select finite "open cover" $\underline{\mathcal{U}}$ of $\underline{B} = (B_n)$ associated with the given finite homotopy trivial open cover $\mathcal{U}$ of B. For each non-empty open set u of B there is a homotopy $H: u \times I \to B$ where $H_0 = i$, the inclusion map and $H_1 = b_*$, the constant map. Extend the homotopy to the closure of the "open set" $\overline{\underline{u}}$ of $\underline{B}$ associated with u then $\underline{i} \simeq \underline{b}_*$. The coherence condition is trivial.

Since ANR's admit homotopy trivial open covers we can state the following:

Proposition-3. *ANR's admit shrinkable open covers.*

We will define "absolute neighborhood strong shape retract."

Definition-4. A compact metric space X embedded in the Hilbert cube $\mathbf{Q}$ is called an absolute neighborhood strong shape retract (ANSSR) if there is a compact ANR $Y \supset X$ in $\mathbf{Q}$ and a family of maps $r_n\colon Y \to X_n$, $n = 1,2,\cdots$ where X_n's are compact ANR-neighborhoods of X and $\cap_n X_n = X$, satisfying the following condition:

for every $\varepsilon > 0$ there is an index $n^*(\varepsilon)$ such that for all $m \geq n \geq n^*(\varepsilon)$ there is a homotopy $H\colon Y \times I \to X_n$ such that

$$H(y,0) = r_n(y)$$

$$H(y,1) = \alpha_{nm} r_m(y),\ \alpha_{nm}\colon X_m \hookrightarrow X_n \text{ is the inclusion map}$$

and

$$d(x,H_t(x)) \text{ for all } x \in X_m,\ t \in I,\ y \in Y.$$

Note that for $t = 0$, $d(x,r_n(x)) = d(i_n(x),r_n i_n(x)) < \varepsilon$ where $i_n\colon X_n \hookrightarrow Y_n$ is the inclusion map.

Remark-4. Clearly, ANR $\Rightarrow$ ANSSR $\Rightarrow$ ANSR (absolute neighborhood shape retract).

ANSSR can also be called a strong FANR. There are compact metric spaces which are not ANR's but are ANSSR's, for example the "Warsaw circle."

The purpose of the definition is the following:

Proposition-4. If a compact metric space X *is an* ANSSR *then it admits a shrinkable open cover.*

Proof. Consider X, Y, X_n's, r_n's etc. as in the definition-4. Since Y is an ANR there is a homotopy trivial open cover $\{u_1, u_2, \cdots, u_\ell\}$ of Y. For $j = 1,2,\cdots,\ell$ write $u_j \cap X = V_j$ and $U_j \cap X_n = W_j^n$, $n = 1,2,\cdots$.

Thus $\{V_1, V_2, \cdots, V_\ell\}$ is a finite open cover of X. For each $n = 1,2,\cdots$ find a finite open cover $\{V_1^n, V_2^n, \cdots V_j^n, \cdots V_\ell^n\}$ of X_n such that $\overline{V}_j^n \subset W_j^n$ and int. $(\cap_n V_j^n)$ = int. $(\cap_n \overline{V}_j^n) = V_j$ where $1 \leq j \leq \ell$. Let ε = min (diameters of V_j) > 0 and $\delta > 0$ be a number such that any two δ-close maps in Y are ε-homotopic. Fix the index j, $(1 \leq j \leq \ell)$ and for simplicity write $U_j = U$, $V_j = V$, $W_j^n = W^n$ and $V_j^n = V^n$.

Let $b \in X$. Then $b \in X_n \subset Y$ for $n = 1,2,\cdots$. Since U is a homotopy trivial open set, there is a homotopy $K\colon U \times I \to Y$ such that $K_o = i\colon U \hookrightarrow Y$, the inclusion map and $K_1 = b_*\colon U \to Y$, the constant map. Then $r_n K|\overline{V}^n$ which we denote by $r_n K$ is a homotopy: $\overline{V}^n \times I \to X_n$ such that $(rK)_0 = r_n\colon \overline{V}^n \hookrightarrow X_n$ and $(r_n K)_1 = (r_n b)_*\colon \overline{V}^n \to X_n$. By the choice of $n \geq n^*(\varepsilon)$, $d(x, r_n(x))$ and $d(b, r_n(b)) < \delta$. Therefore there are linear ε-homotopies $L^n\colon \overline{V}^n \times I \to X_n$ and $M^n\colon \overline{V}^n \times I \to X_n$ such that $L^n(x,0) = x = i_n(x)$, $L^n(x,1) = (r_n K)_0(x) = r_n i_n(x) = r_n(x)$ and $M^n(x,0) = (r_n K)_1(x) = r_n(b)$ and $M^n(x,1) = b$ for $x \in \overline{V}^n$. Let $N^n\colon \overline{V}^n \times I \to X_n$ be the composition of the three homotopies

$$N^n(x,t) = \begin{cases} L^n(x,3t) & 0 \leq t \leq 1/3 \\ r_n K(x,3t-1) & 1/3 \leq t \leq 2/3 \\ M^n(x,3t-2) & 2/3 \leq t \leq 1 \end{cases}$$

Then $N^n(x,0) = L^n(x,0) = i_n(x)$ and

$N^n(x,1) = M^n(x,1) = b$, which is the required homotopy.

Now we will prove the coherence condition. For $m > n \geq n^*(\varepsilon)$, let $N^m\colon \overline{V}^m \times I \to X_m$ be the homotopy between i_m and b_*. Then $\alpha_{nm}N^m\colon \overline{V}^m \times I \to X_n$ is the homotopy between $\alpha_{nm}i_m = i_n$ and $\alpha_{nm}(b_*) = b_*$.

By the definition-4 of ANSSR, there is a homotopy $\hat{H}\colon \alpha_{nm}r_m \simeq r_n\colon Y \times I \to X_n$. Combining with K, we have a homotopy $H\colon \overline{V}^m \times I \times I \to X_n$ such that $H(x,s,0) = \alpha_{nm}r_mK(x,s)$ and $H(x,s,1) = r_nK(x,s)$ for $x \in \overline{V}^m$ and $s \in I$ and $d(x,H_t(x,s)) < \delta$ for all $x \in \overline{V}^m$ and $t \in I$. By the choice of δ, there are linear ε-homotopies $L,M\colon \overline{V}^m \times I \times I \to X_n$ such that

$$L(x,0,t) = x \qquad M(x,0,t) = H(k(x,1),t) = H(x,1,t)$$

$$L(x,1,t) = H(K(x,0),t) = H(x,0,t) \qquad M(x,1,t) = b$$

We can also have $L(x,s,1) = L^n(x,s)$

$$L(x,s,0) = \alpha_{nm}L^m(x,s)$$

and $M(x,s,1) = M^n(x,s)$, $M(x,s,0) = \alpha_{nm}M^m(x,s)$ for $x \in \overline{V}$; $s, t \in I$. Define a homotopy $J\colon \overline{V}^m \times I \times I \to X_n$ by composing the homotopies L, H and M by

$$J(x,s,t) = \begin{cases} L(x,3s,t) & 0 \leq s \leq 1/3 \\ H(x,3s-1,t) & 1/3 \leq s \leq 2/3 \\ M(x,3s-2,t) & 2/3 \leq s \leq 1 \end{cases}$$

Then $J(x,s,0) = \alpha_{nm}N^m(x,s)$

$J(x,s,1) = N^n(x,s)$

$J(x,0,t) = x$ and

$J(x,1,t) = b$ shows that this is the required homotopy.

The author would like to know an answer to the following:

Question-1. Do FANR's admit shrinkable open covers?

The following propositions show that not all compact metric spaces admit a shrinkable open cover.

Proposition-5. *The infinite product of spheres,* $\prod_{i=1}^{\infty} S_i^q (q \geq 1)$ *does not admit a shrinkable open cover.*

Proof. Consider $B_m = \prod_{i=1}^{m} S_i^q$ for $m = 1,2,3,\cdots$ and for $m \geq n$, $\delta_{nm}: B_m \to B_n$ are the projections on the first n factors. Then $\varprojlim (B_n,\delta_{nm}) = \prod_{i=1}^{\infty} S_i^q$. Write $B = \prod_{i=1}^{\infty} S_i^q$. Now every open set u of B is of the form $\prod_{j=1}^{\ell} (\pi_{i_j}^{-1} u_{i_j}) \times \prod_{\substack{i=1 \\ i \neq i_j, 1 \leq j \leq \ell}}^{\infty} S_i^q$ where $\pi_{i_j}: B \to B_{i_j}$ are the projection maps. Suppose for each $j = 1,2,\cdots,\ell$, u_{i_j} is a non-empty homotopy trivial open set such that the inclusion map $i_j: \bar{U}_{i_j} \hookrightarrow B_{i_j}$ is null-homotopic. Then the induced homomorphism $(i_j)_*: \pi_q(\bar{U}_{i_j},b) \to \pi_q(B_{i_j},b)$ is a 0-homomorphism. Consider an "open set" $\underline{u} = (u_n)$ of $\underline{B} = (B_n,\delta_{nm})$ where $u_n = u_{i_j}$ for $n = i_j$, $j = 1,2,\cdots,\ell$ and for $n \neq i_j$ $u_n = S^q$. Then $\underline{\bar{u}} = (\bar{u}_n)$

where $\bar{u}_n = \bar{u}_{i_j}$ or $\bar{u}_n = S^q$. Let $b \in u$, $b = (b_n) \in \underline{B}$ and $\underline{i}: (\underline{\bar{u}},\underline{b}) \hookrightarrow (\underline{B},\underline{b})$ be the inclusion map of ANR-sequences. If $\underline{i}$ is homotopic to the constant map $\underline{b}_*$ then the induced map of the pro-homotopy groups $\underline{i}_*: \underline{\pi}_q(\underline{\bar{u}},\underline{b}) \hookrightarrow \underline{\pi}_q(\underline{B},\underline{b})$ would be the 0-homomorphism. But there is a cofinal subsequence of $\underline{\bar{u}} = (\bar{u}_n)$ where $n \neq i_j$ such that $\underline{i}_*: \underline{\pi}_q(\underline{\bar{u}},\underline{b}) \to \underline{\pi}_q(\underline{B},\underline{b})$ is an isomorphism. Since $\underline{\pi}_q(\underline{B},\underline{b}) \neq 0 \cong \underline{\pi}_q(\underline{\bar{u}},\underline{b})$, $\underline{i}_*$ is not a 0-homomorphism. Thus the "open set" u cannot be a strongly shape trivial. This proves that $B = \prod_{i=1}^{\infty} S_i^q$ cannot admit a shrinkable open cover.

Proposition-6. The dyadic solenoid does not admit a shrinkable open cover.

Proof. For $n = 1,2,\cdots$ let $B_n = S^1$ where $S^1 = \{Z \in \mathbb{C} \mid |Z| = 1\}$ the unit circle and for all $m > n$, $\delta_{nm}: B_m \to B_n$ be the map of degree 2. Then $\varprojlim \underline{B} = \varprojlim (B_n,\delta_{nm}) = B$, the dyadic solenoid. Suppose B admits a shrinkable open cover $\{u_1,u_2,\cdots u_\ell\}$. Then there are open sets $\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_\ell$ of $\underline{B}$ such that for each $j = 1,2,\cdots,\ell$, $u_j = \text{int.}(\varprojlim \underline{u}_j)$ and the inclusion map $\underline{i}: \underline{u}_j \hookrightarrow \underline{B}$ of ANR-sequences is homotopic to a constant map $\underline{b}_*: \underline{u}_j \to \underline{B}$ for $\underline{b} \in \underline{B}$, $\varprojlim \underline{b} = b \in B$. Consider a point $\underline{\pi} = (\pi,\frac{\pi}{2},\cdots,\frac{\pi}{2^{n-1}},\cdots)$ in B where $\pi = -1 \in S^1$ and for $n = 1,2,\cdots$ $\frac{\pi}{2^{n-1}} \in B_n$. Suppose $\underline{\pi} \in \underline{u}_j = (u_j^n)$. Since the inclusion map $\underline{i}: \underline{u}_j \hookrightarrow \underline{B}$ is homotopic to $\underline{\pi}_*: \underline{u}_j \to \underline{B}$, without loss of generality we can assume that for each open set the inclusion map $i_n: u_j^n \hookrightarrow B_n$ is homotopic to the constant map $(\frac{\pi}{2^{n-1}})_*: u_j^n \to B_n$. Since $\underline{\pi}$ lies in the finite number of open sets $\underline{u}_1,\underline{u}_2,\cdots,\underline{u}_\ell$, for each number n there is a number

$m > n$ such that the points $\frac{\pi}{2^{m-1}}$ and $\frac{\pi}{2^{m-1}} + \pi$ lie in the same open set u_j^m. Thus one can find a cofinal subsequence of $\underline{B} = (B_n)$ such that for each index n, the points $\frac{\pi}{2^{n-1}}$ and $\frac{\pi}{2^{n-1}} + \pi$ lie in the open set u_j^n. Since for each n of the cofinal subsequence the inclusion map i_n: $u_j^n \hookrightarrow B_n$ is homotopic to $(\frac{\pi}{2^{n-1}})_*$, there is a path ω_n from $\frac{\pi}{2^{n-1}}$ to $\frac{\pi}{2^{n-1}} + \pi$ which lies in element [0] of the fundamental group $\pi_1(S^1, \frac{\pi}{2^{n-1}})$. But then the homotopy class of the map $\delta_{(n-1)n} \cdot \omega_n \notin [0] \in \pi_1(S^1, \frac{\pi}{2^{n-2}})$, meaning the map $\delta_{(n-1)n}\omega_n$ is not null homotopic, while the inclusion map i_{n-1}: $u_{n-1} \hookrightarrow B_{n-1}$ is null homotopic. This shows that the coherence condition is not satisfied. Thus the dyadic solenoid does not admit a shrinkable open cover.

IV. Main Theorem

Now we will prove the main theorem.

Theorem 1. *Let* $f: p \to q$ *be a shape map between two shape fibrations* $p: E \to B$ *and* $q: E' \to B$ *over* B *where* p *is movable. If* B *admits a shrinkable open cover then* f *is a fiber shape equivalence if and only if* $f|\overset{-1}{p}(b)$ *is a shape equivalence for a point* b *in each strong shape path component of* B.

Proof. We need only to prove that f is a fiber shape equivalence if $f|\overset{-1}{p}(b)$ is a shape equivalence for a point b in each strong shape path component of B.

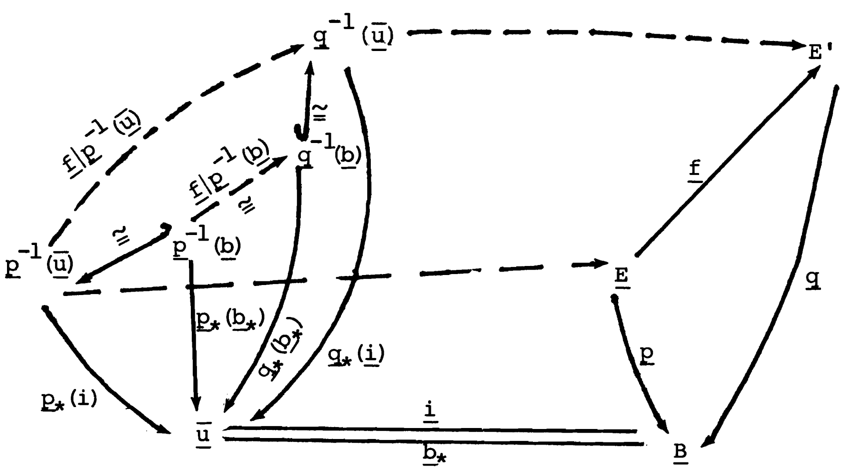
Let $\underline{B} = (B_n, \delta_{nm})$ be an ANR-sequence with $\varprojlim \underline{B} = B$. Let $\mathcal{U} = \{u_1, u_2, u_3, \cdots, u_n\}$ be a strongly shape trivial non-empty open cover of B and $\underline{\mathcal{U}} = \{\underline{u}_1, \underline{u}_2, \underline{u}_3, \cdots \underline{u}_n\}$ be an "open cover" of $\underline{B}$ associated with $\mathcal{U}$; that is $B = \cup_{j=1}^{n} u_j$ and for each $j = 1,2,3,\cdots,n$, $u_j = \mathrm{int}(\lim_{\leftarrow} \underline{u}_j)$.

By the definition of shrinkable open cover, the inclusion level map $\underline{i}_j \colon \underline{\bar{u}} \hookrightarrow \underline{B}$ is strongly homotopic to the constant level map $\underline{b}_{j_*} \colon \underline{\bar{u}}_j \to \underline{B}$ of ANR-sequences for each $j = 1,2,\cdots,n$.

To avoid notational complications, we will drop the suffix j and write $\underline{i} \simeq \underline{b}_* \colon \underline{\bar{u}} \hookrightarrow \underline{B}$.

Let $\underline{p} \colon \underline{E} \to \underline{B}$ and $\underline{q} \colon \underline{E}' \to \underline{B}$ be level maps of ANR-sequences with limit maps p and q and $\underline{f} \colon \underline{E} \to \underline{E}'$ be a shape map of ANR-sequences over B.

Consider the following diagram-1.



Since $\underline{i} \simeq \underline{b}_*$, by the Theorem-B the induced shape fibrations are fiber homotopy equivalent; $\underline{p}_*(\underline{i}) \cong \underline{p}_*(\underline{b}_*)$ and $\underline{q}_*(\underline{i}) \cong \underline{q}_*(\underline{b}_*)$. Also by the assumption $f|p^{-1}(b)$ is a fiber shape equivalence, $\underline{p}_*(\underline{b}_*) \cong \underline{q}_*(\underline{b}_*)$ via the map $\underline{f}|\underline{p}^{-1}(\underline{b})$.

Thus $\underline{f}|\underline{p}^{-1}(\bar{\underline{u}})$: $\underline{p}_*(\underline{i}) \simeq \underline{q}_*(\underline{i})$ is a fiber homotopy equivalence. Since this is true for any "open set" $\underline{u}$ of $\underline{B}$, by the theorem-C, $\underline{f}$: $\underline{p} \to \underline{q}$ is a fiber homotopy equivalence. Hence f: p → q is a fiber shape equivalence.

Corollary 1. *If a space* B *admits a shrinkable open cover then every cell-like, shape fibration* p: E → B *over* B *is a fiber shape equivalence.*

Proof. Let p: E → B be a cell-like shape fibration. We can consider p: E → B as a shape map over B between two shape fibrations p and 1_B: B → B. Note that 1_B is movable. Since p is a cell-like map, $p|p^{-1}(b)$ is a shape equivalence for any b ∈ B. By the Theorem-1, p is a fiber shape equivalence.

Remark. Edwards and Hastings [3] have constructed an example of a cell-like shape fibration p: E → B which fails to be a shape equivalence. However, the base space B is an infinite product of spheres and by the proposition 5-B does not admit a shrinkable open cover. This example shows that in the Theorem-1 the condition on B cannot be omitted.

In [5] the author has proved that every fiber shape equivalence is a hereditary shape equivalence and by [9], every hereditary shape equivalence is a cell-like shape fibration, so it is a cell-like shape fibration. Combining this result with Corollary-1, we have

Corollary 2. *If a space* B *admits a shrinkable open cover then for a map* p: E → B *from a compactum* E *onto* B*, the following three notions are equivalent.*

(1) *cell-like shape fibration*

(2) *fiber shape equivalence*

(3) *hereditary shape equivalence*

Remark. The Corollary-2 answers the Question-3 of T. B. Rushing [10]. The Question-3 is "can one give a reasonable sufficient condition for a shape fibration to be a hereditary shape equivalence?"

The author would like to know an answer to the following:

Question 2. Is there a cell-like shape fibration $p: E \to B$ from a compactum E onto the dyadic solenoid B, which is not a shape equivalence?

Addendum. The author came to know that H. Kato has proved similar results using techniques of strong shape theory (fiber shape categories, Tsukuba J. Math., Vol. 5, #2 (1981), 237-265).

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The William Paterson College of New Jersey

Wayne, New Jersey 07470